RELATIVISTIC CONSERVATION LAWS ON CURVED BACKGROUNDS AND THE THEORY OF COSMOLOGICAL PERTURBATIONS*

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ABSTRACT

We first consider the Lagrangian formulation of general relativity for perturbations with respect to a background spacetime. We show that by combining Noether's method with Belinfante's "symmetrization" procedure we obtain conserved vectors that are independent of any divergence added to the perturbed Hilbert Lagrangian. We also show that the corresponding perturbed energy-momentum tensor is symmetrical and divergenceless but only on backgrounds that are "Einstein spaces" in the sense of A.Z. Petrov. de Sitter or anti-de Sitter and Einstein "spacetimes" are Einstein spaces but in general Friedmann-Robertson-Walker spacetimes are not. Each conserved vector is a divergence of an anti-symmetric tensor, a "superpotential". We find superpotentials which are a generalization of Papapetrou's superpotential and are rigorously linear, even for large perturbations, in terms of the inverse metric density components and their first order derivatives. The superpotentials give correct globally conserved quantities at spatial infinity. They resemble Abbott and Deser's superpotential, but give correctly the Bondi-Sachs total four-momentum at null infinity.

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Next we calculate conserved vectors and superpotentials for perturbations of a Friedmann-Robertson-Walker background associated with its 15 conformal Killing vectors given in a convenient form. The integral of each conserved vector in a finite volume V at a given conformal time is equal to a surface integral on the boundary of V of the superpotential. For given boundary conditions each such integral is part of a flux whose total through a closed hypersurface is equal to zero. For given boundary conditions on V, the integral can be considered as an "integral constraint" on data in the volume and this data always includes the energy-momentum perturbations. We give explicitly these 15 integral constraints and add some simple applications of interest in cosmology. Of particular interest are Traschen integral constraints in which the volume integral contains only the matter energy-momentum tensor perturbations and not the field perturbations. We show that these particular integral constraints are associate with time dependent linear combinations of conformal Killing vectors. Such linear combinations are neither Killing vectors nor conformal Killing vectors. We also find that if we add the "uniform Hubble constant hypersurface" gauge condition of Bardeen, there exists 14 such integral constraints. The exception is associated with conformal time translations $(k = \pm 1)$ or conformal time accelerations (k = 0). As an example we find the constants of motion of a spacetime that is asymptotically Schwarzschild-de Sitter (k = 0).

1. Introduction

(i) Conservation laws and cosmology.

Conservation laws associated with "symmetric" infinitesimal displacements in Friedman-Robertson-Walker spacetimes have been used in relativistic cosmology on several occasions. Infinitesimal displacements are characterized by vector fields and, as we shall see, the vectors used in some applications are not always Killing vectors nor even conformal Killing vectors.

An example in which no Killing nor conformal Killing vectors are used has been given by Traschen [1] who introduced "integral constraints" in terms of "integral constraints vectors". Traschen and Eardley [2] analyzed measurable effects of the cosmic background radiation due to spatially localized perturbations. By using "integral constraints" they pointed to an important reduction of the Sachs-Wolfe [3] effect on the mean square angular fluctuations at large angles of the cosmic background temperature due to local inhomogeneities. Traschen's integral constraints vectors have a somewhat intriguing origin [4]. The equations for integral constraint vectors have been studied by Tod [5]. He showed that these equations are conditions for a spacelike hypersurface to be embeddable in a spacetime with constant curvature of which the solutions are Killing vectors. In Katz, Bičák and Lynden-Bell [6], a paper referred to as KBL97, integral constraints appear as conservation laws with Killing vectors in a de Sitter background; more on this below.

Local differential conservation laws, rather than global ones, have been used by Veeraraghavan and Stebbin [7]. They found and used a conserved "energy-momentum" pseudotensor in an effort to integrate Einstein's equations with scalar perturbations and topological defects in the limit of long wavelengths on a Friedmann-Robertson-Walker spacetime with flat spatial sections (t = const, k = 0). Uzan, Deruelle and Turok [8] realized that these conservation laws might be associated with the conformal Killing vector of time translations and they extended Veeraraghavan and Stebbin's method to Friedmann-Robertson-Walker perturbed spactimes with non-flat spatial sections ($k = \pm 1$). More on this in section 4.

In Lynden-Bell, Katz and Bičák's [9] study of Mach's principle from the relativistic constraint equations, conservation laws yield a general proof that the total angular momentum (and the total of any conserved perturbation of the current which deriving from a "superpotential") must be zero in any closed universe.

As a final example we mention KBL97's analysis of the globally conserved quantities that result from mapping a Friedmann-Robertson-Walker perturbed spacetime on a de Sitter background with its ten Killing vectors.

With these different examples in mind, it made good sense to study the properties and physical interpretation of conservation laws and their superpotentials, in the context of relativistic cosmology, associated with arbitrary displacements in a background as was done in KBL97. In fact, the theory has wider applicability than relativistic cosmology since the background may be any spacetime and there are plenty of examples in general relativity in which backgrounds are used.

(ii) Næther's method and its problems.

KBL97 used the fairly standard method of Noether (see for instance Landau and Lifshitz [10]) to derive conservation laws from the Lagrangian $\hat{\mathcal{L}}_G$ of the perturbations of the gravitational field**

$$\hat{\mathcal{L}}_G = \hat{\mathcal{L}} - \overline{\hat{\mathcal{L}}}, \quad \hat{\mathcal{L}} = -\frac{1}{2\kappa} (\hat{R} + \partial_\mu \hat{k}^\mu), \quad \overline{\hat{\mathcal{L}}} = -\frac{1}{2\kappa} \overline{(\hat{R} + \partial_\mu \hat{k}^\mu)}, \quad \kappa = \frac{8\pi G}{c^4}. \quad (1.1)$$

Here a $\hat{}$ means multiplication by $\sqrt{-g}$, a bar referees to the background, R is the scalar curvature and \hat{k}^{μ} is some vector density. Noether's method associates a conserved vector density \hat{I}^{μ} with any vector ξ^{μ} that generates small displacements. It applies to perturbation theory on any background and provides a "canonical" energy-momentum tensor perturbation. Moreover, the conserved vector is always [11] the divergence of an antisymmetric tensor, a superpotential, $(\hat{I}^{\mu} = \partial_{\nu} \hat{I}^{\mu\nu}, \hat{I}^{\mu\nu} = -\hat{I}^{\nu\mu})$. This has the great practical advantage that integrals of complicated \hat{I}^{μ} 's in a volume V are equal to often much simpler integrals of $\hat{I}^{\mu\nu}$'s on the boundary of V. Noether's method is the most direct and easy way to construct superpotentials, field energy tensors and conserved vector densities with arbitrary backgrounds and for arbitrary ξ^{μ} though the same results can of course be worked out directly from the perturbed Einstein equations. But, at least in our case, this is far more complicated than with the method developed here as we shall see.

Noether's method has, however, two unsatisfactory features. First the Lagrangian density is not unique. A divergence $\partial_{\mu}\hat{k}^{\mu}$ can and must be added to the Hilbert Lagrangian because the latter leads [12] to Komar's [13] conservation law which gives the wrong mass to angular momentum ratio with an "anomalous" factor of two in the weak field limit [14] and does not give the Bondi mass [15] at null infinity [16]. Divergences are also added to comply with different boundary conditions. Various divergences have thus been added to \hat{R} for different reasons. Møller [17], using a tetrad representation e^{α}_{μ} with $g_{\mu\nu} = \eta_{\alpha\beta}e^{\alpha}_{\mu}e^{\beta}_{\nu}$, would have taken a $\hat{k}^{\mu} = \hat{g}^{\mu\nu}g^{\rho\sigma}(e^{\alpha}_{\rho}D_{\sigma}e^{\beta}_{\nu} - e^{\alpha}_{\nu}D_{\rho}e^{\beta}_{\sigma})\eta_{\alpha\beta}$. York [18], using a foliation, would have chosen $\hat{k}^{\mu} = 2\left(\varepsilon n^{\mu}D_{\nu}\hat{n}^{\nu} - n^{\nu}D_{\nu}\hat{n}^{\mu}\right)$ with n^{μ} $(n^{\mu}n_{\mu} = \varepsilon = \pm 1)$ the normal vectors of his closed hypersurfaces. KBL97 wanted a field energy tensor quadratic in first order derivatives and took therefore like Rosen [19] a long time before

$$\hat{k}^{\mu} = \frac{1}{\sqrt{-g}} \overline{D}_{\nu} (-gg^{\mu\nu}). \tag{1.2}$$

^{**} Notations are properly defined in section 2. Here we assume the reader to be familiar with current notations.

 \overline{D}_{μ} is a covariant derivation with respect to the background metric $\overline{g}_{\mu\nu}$. Second, the canonical field energy momentum tensor is not symmetrical nor is it divergenceless. On a flat background, the energy-momentum tensor is divergenceless but is still not symmetrical and the angular momentum is not conserved; it does not include the helicity of the field. It thus appears that conservation laws obtained by Noether's method have an unsatisfactory weak field limit on a flat background at least as far as angular momentum is concerned.

To remedy that situation we suggest in this paper to modify Noether conserved vectors using Belinfante's [20] trick in classical field theory. It is an easy matter to adapt his method to perturbation theory on curved backgrounds. Belinfante's modification leads to energy-momentum tensors which ensures, at least in classical field theory, that angular momentum includes the helicity and is then conserved. The Belinfante trick has been applied by Papapetrou [21] to general relativity in an effort to calculate the total angular momentum at spatial infinity.

These new conserved vectors have none of the drawbacks just described and in addition have very appealing new properties.

(iii) A summary of theoretical results.

It may be useful, at this stage, to give a summary of our main theoretical results, we mean those valid in general, not only in relativistic cosmology.

(a) We find that there exist a conserved vector density $\hat{\mathcal{I}}^{\mu}$ associated with any vector generating infinitesimal displacements and which is the divergence of a superpotential $\hat{\mathcal{I}}^{\mu\nu}$; it is of the following form

$$\hat{\mathcal{I}}^{\mu} = \hat{\mathcal{T}}^{\mu}_{\nu} \xi^{\nu} + \hat{\mathcal{Z}}^{\mu} = \partial_{\nu} \hat{\mathcal{I}}^{\mu\nu}. \tag{1.3}$$

 $\hat{\mathcal{T}}^{\mu}_{\nu}$ represents a matter plus field energy-momentum tensor density perturbation relative to the background. $\hat{\mathcal{Z}}^{\mu}$ is a vector density that is only equal to zero if ξ^{ν} is a Killing vector of the background, which we denote then by $\bar{\xi}^{\nu}$. As a consequence of Eq. (1.3) a volume integral F of $\hat{\mathcal{I}}^{\mu}$ equal the surface integral of $\hat{\mathcal{I}}^{\mu\nu}$ on its boundary S

$$F = \int_{V} \hat{\mathcal{I}}^{\mu} dV_{\mu} = \oint_{S} \hat{\mathcal{I}}^{\mu\nu} dS_{\mu\nu}$$
 (1.4)

and the total flux F through a closed hypersurface V is equal to zero. That is what is meant $\hat{\mathcal{I}}^{\mu}$ being a conserved current. Equalities such as (1.4) may be regarded as "integral constraints" in the following sense. Suppose that boundary values on S and thus $\hat{\mathcal{I}}^{\mu\nu}$ are known. Then Eq. (1.4) represents constraints on the perturbations of the energy-momentum tensor which is always part of $\hat{\mathcal{T}}^{\mu}_{\nu}$. There are thus as many integral constraints as there are displacement vectors ξ^{μ} (but not all are equally interesting). Our notion of "integral constraints" is slightly different from that introduced by Traschen [1]. She calls integral constraint an expression like Eq. (1.10) below in which the boundary term

is equal to zero and the volume integral contains only the perturbation of the matter energy-momentum tensor. One may wish to see in Eq. (1.4) a definition of quasi-local conservation laws if the boundary is not at the border of spacetime itself.

(b) The conserved vector, and thus the corresponding \hat{T}^{μ}_{ν} are independent of any divergence added to the Hilbert Lagrangian density of the perturbations. It has been noticed before by Bak, Cangemi and Jackiw [22] that Belinfante's modification of the Noether currents obtained from Hilbert's or Einstein's Lagrangians lead to the same symmetric and divergenceless energy-momentum tensor relative to a flat background in Minkowski coordinates. Ours is a generalization of this finding for any divergence added to the Hibert Lagrangian, for arbitrary perturbations with respect to any background in arbitrary coordinates.

We want to stress that since our conserved vectors are independent of an added divergence, they are also *independent* of boundary conditions. This result is in line with classical field ideas. The opposite view that pseudotensors and superpotentials must depend on boundary conditions has been held for instance in [23].

(c) From Eq. (1.3) follows that for each Killing vector of the background $\bar{\xi}^{\nu}$ there exists a conserved vector $\hat{\mathcal{J}}^{\mu}$ and a corresponding superpotential $\hat{\mathcal{J}}^{\mu\nu}$ such that

$$\hat{\mathcal{J}}^{\mu} = \hat{T}^{\mu}_{\nu} \bar{\xi}^{\nu} = \partial_{\nu} \hat{\mathcal{J}}^{\mu\nu}. \tag{1.5}$$

This expression looks also very much like a conservation law in classical field theory.

- (d) $\hat{T}^{\mu\nu} = \hat{T}^{\mu}_{\rho} \overline{g}^{\rho\nu}$ is symmetrical and divergenceless if and only if the background is an Einstein space in the sense of A.Z. Petrov [24], that is if $\overline{R}_{\mu\nu} = -\overline{\Lambda} \overline{g}_{\mu\nu}$ where $\overline{\Lambda}$ is necessarily a constant. de Sitter spacetimes belong to that category. Other Friedmann-Robertson-Walker spacetimes that are currently used in cosmology do not.
- (e) The new superpotential $\hat{\mathcal{I}}^{\mu\nu}$ is reminiscent of many well known ones (see section 2 for details) and has a simple form:

$$\hat{\mathcal{I}}^{\mu\nu} = -\hat{\mathcal{I}}^{\nu\mu} = \frac{1}{\kappa} \hat{l}^{\sigma[\mu} \overline{D}_{\sigma} \xi^{\nu]} + \frac{1}{\kappa} \overline{D}_{\sigma} \left(\hat{l}^{\rho[\mu} \overline{g}^{\nu]\sigma} - \hat{l}^{\sigma[\mu} \overline{g}^{\nu]\rho} \right) \xi_{\rho}. \tag{1.6}$$

In this expression $\hat{l}^{\mu\nu}$ is the perturbed inverse metric density:

$$\hat{l}^{\mu\nu} = \hat{g}^{\mu\nu} - \overline{\hat{g}^{\mu\nu}}.\tag{1.7}$$

The superpotential has the remarkable property of being linear in $\hat{l}^{\mu\nu}$. Linearity is a valuable property; the linear approximation is not different from the non linear one. Global exact conservation quantities of know asymptotic fields with unknown sources can be calculated and given physical meaning. The superpotential (1.6) satisfies standard criteria of global conservation laws in asymptotically flat spacetimes at spatial and at null infinity

(see appendix). Also second order corrections of the energy-momentum tensor due to field energy contributions are readily calculable from our formulas.

Those are the principal theoretical results of the paper.

(iv) The 15 Conformal Killing vectors of cosmological backgrounds and their associated conservation laws and integral constraints.

To illustrate our new conservation laws in theoretical cosmology we consider the conserved vectors and superpotentials associated with the 15 conformal Killing vectors of Friedmann-Robertson-Walker spacetimes. By conformal Killing vectors we mean the 15 linearly independent solutions of the conformal Killing vector equations

$$\overline{D}_{(\mu}\xi_{\nu)} = \frac{1}{4}\overline{g}_{\mu\nu}\overline{D}_{\rho}\xi^{\rho}, \qquad \xi_{\nu} = \overline{g}_{\nu\rho}\xi^{\rho}. \tag{1.8}$$

The 15 conformal Killing vectors include by definition the 6 pure Killing vectors for which $\overline{D}_{\rho}\xi^{\rho}=0$. Friedmann-Robertson-Walker spacetimes are conformal to Minkowski's spacetime. There are thus similarities between the 4 Killing vectors of translations, the 3 rotations, 3 Lorentz boosts, 3 center of mass position, 1 dilatation and the 4 "accelerations" of Minkowski's spacetime. Such similarities are helpful in geometrical interpretations. A presentation of those conformal Killing vectors of Minkowski spacetime in a form that appeals to physicists is given in Fulton, Rohrlich and Witten [25] who have a particular liking for the accelerations which they explain well and to which we referee the reader interested in those slightly unfamiliar conformal Killing vectors .

Now follows a brief summary of the results.

(a) We give the 15 linearly independent solutions of Eq. (1.8) in a simple mathematical form. We are not interested in the algebra of the conformal group. We are mostly interested here in quasi-local or in global conservation laws or integral constraints for volumes in a sphere (parametrized by r) at a given instant of conformal time η . Thus, in (η, x^k) coordinates we are looking for integrals of the form

$$\int_{r} \mathcal{I}^{0} dV = \oint_{r} \mathcal{I}^{0l} dS_{l}. \tag{1.9}$$

Any linear combination of such integrals with time dependent coefficients are still of the same form. We give what seem to us the 15 simplest linear combinations. Notice that linear combinations of conformal Killing vectors with time dependent coefficients are not conformal Killing vectors anymore. Notice also that the left hand integrand of such linear combinations is not a zero component of a conserved vector anymore.

(b) We next turn our attention to those linear combinations in which the volume integrand depends only on the matter energy-momentum perturbations δT_{μ}^{0} thus of the form

$$\int_{V} \delta T_{\nu}^{0} V^{\nu} dV = \oint_{S} B^{l} dS_{l}. \tag{1.10}$$

There are 10 integral constraints of this form, 6 are associated with the pure Killing vectors of the Friedmann-Robertson-Walker spacetimes and 4 are the vectors found by Traschen [1]. Thus Traschen's "integral constraint vectors" appear here as linear combinations of conformal Killing vectors with time dependent coefficients.

- (c) We then show that if we apply the uniform Hubble expansion gauge studied by Bardeen [26] all but one of the 15 conformal Killing vectors are associated with integral conservation laws of the form (1.10). The exception is associated with conformal time translations if $k = \pm 1$ or conformal time accelerations if k = 0. A look at Eq. (1.10) shows that these integrals might be constructed directly from Einstein's constraint equations. However the V^{μ} 's though simple are not all that easy to guess as we shall see. The integral constraints have often simple geometrical interpretations by analogy with classical mechanics. With conformal Killing vectors they are momenta of order 0, 1 or 2.
- (d) We also give a non-trivial example of globally conserved quantities on a background that is not asymptotically flat. We calculate the constants of motion of spacetimes that are asymptotically Schwarzschild-de Sitter (with k=0). Because of the high degree of symmetry of the asymptotic conditions we find 13 globally conserved quantities equal to zero, that is 13 Traschen's like integral constraints. There are 2 constants of motion which are different from zero.

Such are the examples of cosmological interest studied in this paper.

(v) Presentation of the paper.

In the following section we describe in detail the way to obtain Noether's conservation laws on curved backgrounds. A few parts of that chapter are taken from KBL97 but they are well worth repeating here. Section 3 introduces the Belinfante correction to Noether's conserved vectors. There we show that the modified conserved vectors and their associated superpotentials are unchanged if we add a divergence to the Hilbert Lagrangian for the perturbations. At the end of section 3 we derive Rosenfeld's [27] identities which give beautiful relations among complicated quantities of interest. Rosenfeld's identities are referred to as "cascade equations" by Julia and Silva [28]. In that section we also obtain the theoretical results summarized above in (iii). In section 4 we briefly describe how we found the 15 conformal Killing vectors of Friedmann-Robertson-Walker spacetimes in appropriate coordinates and how to calculate the corresponding conserved vectors and superpotentials in a 1+3 standard decomposition. The same section contains the examples just described in (iv). In section 5 we emphasise the role of superpotentials in the development of conservation law theory for general relativity and make a short review of historically important superpotentials on a flat background. A flat background is quite useful to study conservation laws in general relativity as pointed out by Rosen [19] a long time ago.

Finally in appendix we briefly show that our new superpotential has the normally expected global properties at null and spatial infinity in asymptotically flat spacetimes.

Each section is preceded by a summary which gives the motivations and point out where the reader will find the principal formulas. The body of the sections themselves are written for readers who are interested in the mathematical details. However, most elaborate but straightforward calculations are not given in detail. Unfortunately general relativity is replete with them.

2. Noether

(i) Motivations and summary of results.

In this section we apply Noether's method to the Lagrangian $\hat{\mathcal{L}}_G$ defined in Eq. (1.1) with the vector density \hat{k}^{μ} defined in Eq. (1.2). This is the KBL97 Lagrangian. The conserved vector \hat{I}^{μ} obtained in this way is given in Eq. (2.17). A look at Eq. (2.17) shows that it contains a canonical energy-momentum tensor density $\hat{\theta}^{\mu}_{\nu}$ explicitly written in Eq. (2.18) with \hat{t}^{μ}_{ν} given in Eq. (2.13), an "helicity" term $\hat{\sigma}^{\mu\rho\sigma}$ given by Eq. (2.14) and a vector density $\hat{\eta}^{\mu}$, see Eq. (2.19), defined in terms of the derivatives of the Lie derivatives of the background metrics $\bar{g}_{\rho\sigma}$ or $\bar{z}_{\rho\sigma}$ defined in Eq. (2.11); $\hat{\eta}^{\mu}$ is thus zero if ξ^{μ} is a Killing vector $\bar{\xi}^{\mu}$ of the background.

The canonical energy-momentum tensor density is neither symmetrical nor divergenceless except on a flat background [see Eq. (5.1a)] in which case the canonical field energy-momentum \hat{t}^{μ}_{ν} reduces to Einstein's pseudo-tensor in Minkowski coordinates X^{μ} . In the same coordinates the helicity tensor density $\hat{\sigma}^{\mu[\rho\sigma]}$ is that which was given by Papapetrou [21]. The conserved vector density \hat{I}^{μ} is thus a generalization of standard old results to finite perturbations of a curved background with arbitrary vectors not only with Killing vectors. \hat{I}^{μ} is also written as the divergence of a superpotential $\hat{I}^{\mu\nu}$ which is given in Eq. (2.21) and also by Eq. (2.22).

The present section also contains the energy-momentum tensor, the helicity, and the superpotential that are obtained from the Hilbert Lagrangian density

$$\hat{\mathcal{L}}_G' = -\frac{1}{2\kappa} (\hat{R} - \overline{\hat{R}}). \tag{2.1}$$

The quantities, $\hat{\theta}_{\nu}^{\prime\mu}$ and $\hat{\sigma}^{\prime\mu\rho\sigma}$ are defined by Eqs. (2.24); the superpotential $\hat{K}^{\mu\nu}$ is given by "Eq. (2.21) minus its ξk -term". While $\hat{\theta}_{\nu}^{\mu}$ and, to a certain extend, also $\hat{\sigma}^{\mu\rho\sigma}$ reduce to familiar quantities on a flat background, $\hat{\theta}_{\nu}^{\prime\mu}$ and $\hat{\sigma}^{\prime\mu[\rho\sigma]}$ are much more complicated; $\hat{\theta}_{\nu}^{\prime\mu}$ contains second order derivatives. In fact we gain in clarity and simplicity by starting the calculations with $\hat{\mathcal{L}}_{G}$ rather than $\hat{\mathcal{L}}_{G}^{\prime}$; the end product in section 3 is independent of $\partial_{\mu}\hat{k}^{\mu}$.

The conservation law $\partial_{\mu}\hat{I}^{\mu}=0$ has been studied in KBL97. Therefore the present section is mainly mathematical; it gives the necessary ingredients for sections 3 and 4.

(ii) The conserved vector density \hat{I}^{μ} .

Let $g_{\mu\nu}(x^{\lambda})$ be the metric of the perturbed spacetime \mathcal{M} and $\overline{g}_{\mu\nu}$ be the metric of the background $\overline{\mathcal{M}}$ both with signature -2. Once we have chosen a smooth global mapping such that each point P of \mathcal{M} is mapped on a point \overline{P} of $\overline{\mathcal{M}}$, we can use the convention that \overline{P} and P shall always be given the same coordinates $\overline{x^{\mu}} = x^{\mu}$. This convention implies that coordinate transformations on \mathcal{M} inevitably induce the same coordinate transformations with the same functions on $\overline{\mathcal{M}}$. With this convention, such expressions as $g_{\mu\nu} - \overline{g}_{\mu\nu}$ become true tensors. However if the particular mapping has been left unspecified, we are still free to change it. The form of the equations for perturbations must inevitably contain a gauge invariance corresponding to this freedom.

Let $R^{\lambda}_{\nu\rho\sigma}$ and $\overline{R}^{\lambda}_{\nu\rho\sigma}$ be the curvature tensors of \mathcal{M} and $\overline{\mathcal{M}}$. These are related as follows:

$$R^{\lambda}_{\nu\rho\sigma} = \overline{D}_{\rho} \Delta^{\lambda}_{\nu\sigma} - \overline{D}_{\sigma} \Delta^{\lambda}_{\nu\rho} + \Delta^{\lambda}_{\rho\eta} \Delta^{\eta}_{\nu\sigma} - \Delta^{\lambda}_{\sigma\eta} \Delta^{\eta}_{\nu\rho} + \overline{R}^{\lambda}_{\nu\rho\sigma}. \tag{2.2}$$

Here \overline{D}_{ρ} are covariant derivatives with respect to $\overline{g}_{\mu\nu}$ and $\Delta^{\lambda}_{\mu\nu}$ is the difference between Christoffel symbols in \mathcal{M} and $\overline{\mathcal{M}}$:

$$\Delta^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \overline{\Gamma}^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho} \left(\overline{D}_{\mu}g_{\rho\nu} + \overline{D}_{\nu}g_{\rho\mu} - \overline{D}_{\rho}g_{\mu\nu} \right). \tag{2.3}$$

Our quadratic Lagrangian density $\hat{\mathcal{L}}_G$ for the gravitational field is here defined by Eq. (1.1) with \hat{k}^{μ} given in Eq. (1.2). The caret means, as we said before, multiplication by $\sqrt{-g}$, never by $\sqrt{-\overline{g}}$. Thus, if $\hat{R} = \sqrt{-g}R$, \hat{R} will unambiguously mean $\sqrt{-\overline{g}R}$. Notice that $\hat{R} = \sqrt{-g}R \neq \hat{R} = \sqrt{-\overline{g}R}$. The vector density \hat{k}^{μ} can also be written in the following form that is often useful in calculations:

$$\hat{k}^{\mu} = \frac{1}{\sqrt{-g}} \overline{D}_{\nu} \left(-gg^{\mu\nu} \right) = \hat{g}^{\mu\rho} \Delta^{\sigma}_{\rho\sigma} - \hat{g}^{\rho\sigma} \Delta^{\mu}_{\rho\sigma}, \tag{2.4}$$

 $\partial_{\mu}\hat{k}^{\mu}$ cancels second order derivatives of $g_{\mu\nu}$ in \hat{R} . $\hat{\mathcal{L}}$ is the Lagrangian density used by Rosen [19]. $\overline{\hat{\mathcal{L}}}$ is $\hat{\mathcal{L}}$ in which $g_{\mu\nu}$ has been replaced by $\overline{g}_{\mu\nu}$. When $g_{\mu\nu} = \overline{g}_{\mu\nu}$, $\hat{\mathcal{L}}_G$ is thus identically zero. The following formula, deduced from Eqs. (2.2) and (2.4), shows explicitly how $\hat{\mathcal{L}}_G$ is quadratic in the first order derivatives of $g_{\mu\nu}$ or, equivalently, quadratic in $\Delta^{\mu}_{\rho\sigma}$:

$$\hat{\mathcal{L}}_G = \frac{1}{2\kappa} \hat{g}^{\mu\nu} \left(\Delta^{\rho}_{\mu\nu} \Delta^{\sigma}_{\rho\sigma} - \Delta^{\rho}_{\mu\sigma} \Delta^{\sigma}_{\rho\nu} \right) - \frac{1}{2\kappa} \hat{l}^{\mu\nu} \overline{R}_{\mu\nu}. \tag{2.5}$$

where $\hat{l}^{\mu\nu}$ is the perturbed metric density defined in Eq. (1.7). If the background is flat and denoted $\overline{\mathcal{M}}_0$ and if we use Minkowski coordinates X^{μ} , then $\overline{g}_{\mu\nu}=\eta_{\mu\nu}=diag(1,-1,-1,-1), \ \overline{\Gamma}^{\lambda}_{\mu\nu}=0$ and

$$\hat{\mathcal{L}}_G = -\frac{1}{2\kappa} \left(\hat{R} + \partial_\mu \hat{k}^\mu \right) = \frac{1}{2\kappa} \hat{g}^{\mu\nu} \left(\Gamma^\sigma_{\mu\rho} \Gamma^\rho_{\sigma\nu} - \Gamma^\rho_{\mu\nu} \Gamma^\sigma_{\rho\sigma} \right) \tag{2.6}$$

which is Einstein's [29] Lagrangian. $\hat{\mathcal{L}}_G$ is thus a generalization of Einstein's Lagrangian density to perturbations on a curved background.

Lie differentials are particularly convenient in describing infinitesimal displacements in both \mathcal{M} and $\overline{\mathcal{M}}$; if the mapping was defined before the displacements it remains defined after displacements. Let $\Delta x^{\mu} = \xi^{\mu} \Delta \lambda$ represent an infinitisimal one-parameter displacement generated by a sufficiently smooth vector field ξ^{μ} , the corresponding infinitesimal change in tensors are given in terms of Lie derivatives with respect to this vector field ξ^{μ} , $\Delta g_{\mu\nu} = \pounds_{\xi} g_{\mu\nu} \Delta \lambda$, etc. The Lie derivatives may be written in terms of partial derivatives ∂_{μ} , covariant derivative \overline{D}_{μ} with respect to $\overline{g}_{\mu\nu}$, or covariant derivative D_{μ} with respect to $g_{\mu\nu}$. Thus,

$$\pounds_{\xi} g_{\mu\nu} = g_{\mu\lambda} \partial_{\nu} \xi^{\lambda} + g_{\nu\lambda} \partial_{\mu} \xi^{\lambda} + \xi^{\lambda} \partial_{\lambda} g_{\mu\nu}$$
 (2.7a)

$$= g_{\mu\lambda} \overline{D}_{\nu} \xi^{\lambda} + g_{\nu\lambda} \overline{D}_{\mu} \xi^{\lambda} + \xi^{\lambda} \overline{D}_{\lambda} g_{\mu\nu}$$
 (2.7b)

$$= g_{\mu\lambda} D_{\nu} \xi^{\lambda} + g_{\nu\lambda} D_{\mu} \xi^{\lambda}. \tag{2.7c}$$

Consider now the Lie derivative $\mathcal{L}_{\xi}\hat{\mathcal{L}}$ of $\hat{\mathcal{L}}$ in Eq. (1.1), not of $\hat{\mathcal{L}}_{G}$. The Lie derivative of a scalar density like $\hat{\mathcal{L}}$ is the ordinary divergence $\partial_{\mu}(\hat{\mathcal{L}}\xi^{\mu})$. With the variational principle in mind we can thus write the following identity

$$\pounds_{\xi}\hat{\mathcal{L}} = \frac{1}{2\kappa}\hat{G}^{\mu\nu}\pounds_{\xi}g_{\mu\nu} - \frac{1}{2\kappa}\partial_{\mu}\left(\hat{g}^{\rho\sigma}\pounds_{\xi}\Gamma^{\mu}_{\rho\sigma} - \hat{g}^{\mu\rho}\pounds_{\xi}\Gamma^{\sigma}_{\rho\sigma} + \pounds_{\xi}\hat{k}^{\mu}\right) = \partial_{\mu}\left(\hat{\mathcal{L}}\xi^{\mu}\right) \tag{2.8}$$

where Einstein's tensor density $\hat{G}^{\mu\nu} = \hat{R}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\hat{R}$ is the variational derivative of $2\kappa\hat{\mathcal{L}}$ with respect to $g_{\mu\nu}$. Equation (2.8) is easily converted into a conservation law by treating the first term after the first equality sign as follows: (a) replace $\mathcal{L}_{\xi}g_{\mu\nu}$ by the expression (2.7c), (b) use the contracted Bianchi identities $D_{\nu}G^{\mu\nu} = 0$ and (c) use Einstein's equations $\hat{G}^{\mu}_{\nu} = \kappa \hat{T}^{\mu}_{\nu}$. Identity (2.8) becomes then a conservation law of this form

$$\partial_{\mu}\hat{i}^{\mu} = 0, \quad \text{with} \quad \hat{i}^{\mu} = \hat{T}^{\mu}_{\nu}\xi^{\nu} - \frac{1}{2\kappa} \left(\hat{g}^{\rho\sigma} \pounds_{\xi} \Gamma^{\mu}_{\rho\sigma} - \hat{g}^{\mu\rho} \pounds_{\xi} \Gamma^{\sigma}_{\rho\sigma} + \pounds_{\xi} \hat{k}^{\mu} \right) - \hat{\mathcal{L}}\xi^{\mu}. \tag{2.9}$$

Now comes another exercise which consists in replacing $\pounds_{\xi}g_{\mu\nu}$ and the D-derivatives of $\pounds_{\xi}g_{\mu\nu}$ that appear in $\pounds_{\xi}\Gamma^{\mu}_{\rho\sigma}$ - see Eq. (2.20) below - by \overline{D} -derivatives and \overline{D} \overline{D} -derivatives using Eq. (2.7b) this time. The relation between the two kinds of derivatives, D and \overline{D} is best illustrated on the following simple case

$$D_{\nu}\xi^{\mu} = \partial_{\nu}\xi^{\mu} + \Gamma^{\mu}_{\nu\rho}\xi^{\rho} = \overline{D}_{\nu}\xi^{\mu} + \Delta^{\mu}_{\nu\rho}\xi^{\rho}. \tag{2.10}$$

The relations is: (a) write D-derivatives in terms of ∂ -derivatives and Γ 's (b) replace the ∂ 's by \overline{D} 's and Γ 's by Δ 's. If we operate like that on the terms between parenthesis of \hat{i}^{μ} in Eq. (2.9), we obtain after a tedious but quite straightforward calculation the following

result. \hat{i}^{μ} has a term in ξ^{μ} , one in $\overline{D}_{\rho}\xi_{\sigma}$ and one that contains the derivatives of the Lie derivatives of the background metric or of ***

$$\bar{z}_{\rho\sigma} \equiv \frac{1}{2} \pounds_{\xi} \bar{g}_{\rho\sigma} = \overline{D}_{(\rho} \xi_{\sigma)}. \tag{2.11}$$

Here $\xi_{\sigma} = \overline{g}_{\sigma\mu}\xi^{\mu}$. Indices will always been displaced with the background metric $\overline{g}_{\mu\nu}$, never with $g_{\mu\nu}$. Thus \hat{i}^{μ} has this form

$$\hat{i}^{\mu} = \left(\hat{T}^{\mu}_{\nu} + \frac{1}{2\kappa}\hat{g}^{\rho\sigma}\overline{R}_{\rho\sigma}\delta^{\mu}_{\nu} + \hat{t}^{\mu}_{\nu}\right)\xi^{\nu} + \hat{\sigma}^{\mu\rho\sigma}\overline{D}_{\rho}\xi_{\sigma} + \hat{e}^{\mu}. \tag{2.12}$$

The undefined symbols in Eq. (2.12) satisfy the following equalities,

$$2\kappa \hat{t}^{\mu}_{\nu} = \hat{g}^{\rho\sigma} \left[\left(\Delta^{\lambda}_{\rho\lambda} \Delta^{\mu}_{\sigma\nu} + \Delta^{\mu}_{\rho\sigma} \Delta^{\lambda}_{\lambda\nu} - 2\Delta^{\mu}_{\rho\lambda} \Delta^{\lambda}_{\sigma\nu} \right) - \delta^{\mu}_{\nu} \left(\Delta^{\eta}_{\rho\sigma} \Delta^{\lambda}_{\eta\lambda} - \Delta^{\eta}_{\rho\lambda} \Delta^{\lambda}_{\eta\sigma} \right) \right] + \hat{g}^{\mu\lambda} \left(\Delta^{\sigma}_{\rho\sigma} \Delta^{\rho}_{\lambda\nu} - \Delta^{\sigma}_{\lambda\sigma} \Delta^{\rho}_{\rho\nu} \right), \tag{2.13}$$

$$2\kappa\hat{\sigma}^{\mu\rho\sigma} = (g^{\mu\rho}\overline{g}^{\sigma\nu} + \overline{g}^{\mu\sigma}g^{\rho\nu} - g^{\mu\nu}\overline{g}^{\rho\sigma})\hat{\Delta}^{\lambda}_{\nu\lambda} - (g^{\nu\rho}\overline{g}^{\sigma\lambda} + \overline{g}^{\nu\sigma}g^{\rho\lambda} - g^{\nu\lambda}\overline{g}^{\rho\sigma})\hat{\Delta}^{\mu}_{\nu\lambda}$$
(2.14)

and

$$2\kappa\hat{e}^{\mu} = \hat{g}^{\mu\lambda}\partial_{\lambda}\bar{z} + \hat{g}^{\rho\sigma}\left(\overline{D}^{\mu}\bar{z}_{\rho\sigma} - 2\overline{D}_{\rho}\bar{z}^{\mu}_{\sigma}\right), \qquad \bar{z} = \overline{g}^{\rho\sigma}\bar{z}_{\rho\sigma} = \overline{D}_{\lambda}\xi^{\lambda}. \tag{2.15}$$

Had we applied Eq. (2.8) to $\overline{\hat{\mathcal{L}}}$ instead of $\hat{\mathcal{L}}$, we would have written everywhere $\overline{g}_{\mu\nu}$ instead of $g_{\mu\nu}$, from Eq. (2.8) up to Eq. (2.15). We would have found barred, conserved vector densities $\overline{\hat{i}^{\mu}}$ instead of \hat{i}^{μ} that are as follows:

$$\overline{\hat{i}^{\mu}} = \left(\overline{\hat{T}^{\mu}_{\nu}} + \frac{1}{2\kappa}\overline{\hat{R}}\delta^{\mu}_{\nu}\right)\xi^{\nu} + \overline{\hat{e}^{\mu}}.$$
(2.16)

The simpler form of Eq. (2.16) compared with Eq. (2.12) comes from the fact that $\overline{\Delta_{\rho\sigma}^{\mu}} = 0$ and thus $\hat{t}_{\nu}^{\mu} = \overline{\hat{\sigma}^{\mu\rho\sigma}} = 0$. Conserved vector densities for $\hat{\mathcal{L}}_{G} = \hat{\mathcal{L}} - \overline{\hat{\mathcal{L}}}$ are thus obtained by subtracting Eq. (2.16) from Eq. (2.12). We find in this way, a conserved vector density relative to the background \hat{I}^{μ} :

$$\hat{I}^{\mu} = \hat{i}^{\mu} - \overline{\hat{i}^{\mu}} = \hat{\theta}^{\mu}_{\nu} \xi^{\nu} + \hat{\sigma}^{\mu\rho\sigma} \overline{D}_{\rho} \xi_{\sigma} + \hat{\eta}^{\mu}, \qquad \partial_{\mu} \hat{I}^{\mu} = 0, \tag{2.17}$$

with

$$\hat{\theta}^{\mu}_{\nu} = \hat{T}^{\mu}_{\nu} - \overline{\hat{T}^{\mu}_{\nu}} + \frac{1}{2\kappa} \hat{l}^{\rho\sigma} \overline{R}_{\rho\sigma} \delta^{\mu}_{\nu} + \hat{t}^{\mu}_{\nu}, \tag{2.18}$$

^{***} The presence of $\bar{z}_{\rho\sigma}$ comes from replacing second derivatives using the following identity $\overline{D}_{\rho\sigma}\xi^{\mu} = \bar{R}^{\mu}_{\ \sigma\rho\nu}\xi^{\nu} + 2\overline{D}_{(\rho}\bar{z}^{\mu}_{\sigma)} - \overline{D}^{\mu}\bar{z}_{\rho\sigma}$.

and

$$\hat{\eta}^{\mu} = \hat{e}^{\mu} - \overline{\hat{e}^{\mu}} = \frac{1}{2\kappa} \left[\hat{l}^{\mu\lambda} \partial_{\lambda} \bar{z} + \hat{l}^{\rho\sigma} \left(\overline{D}^{\mu} \bar{z}_{\rho\sigma} - 2 \overline{D}_{\rho} \bar{z}^{\mu}_{\sigma} \right) \right]. \tag{2.19}$$

For Killing vectors $\bar{\xi}^{\mu}$, $\hat{\eta}^{\mu} = 0$ and in Eq. (2.17), $\hat{\sigma}^{\mu\rho\sigma}\overline{D}_{(\rho}\bar{\xi}_{\sigma)}$ is equally zero. The remaining part of the σ -term contains the antisymmetric part $\hat{\sigma}^{\mu[\rho\sigma]}$ to which we refereed in the beginning of this section. It plays the role of a (relative) helicity in linearized quantum gravity [30] and, see subsection 3(iii) below, is similar to the helicity in electromagnetism [31].

(iii) The superpotential.

Had we not replaced $(1/\kappa)\hat{G}^{\mu}_{\nu}$ by \hat{T}^{μ}_{ν} in Eq. (2.9) the conservation law $\partial_{\mu}\hat{i}^{\mu}=0$ would have remained an identity instead of becoming an equation. Thus \hat{i}^{μ} must be equal to the divergence of an antisymmetric tensor density or "superpotential" [32] for any $g_{\mu\nu}$, $\overline{g}_{\mu\nu}$ and ξ^{μ} , i.e., there exists a $\hat{i}^{\mu\nu}=-\hat{i}^{\nu\mu}$ such that $\hat{i}^{\mu}=\partial_{\nu}\hat{i}^{\mu\nu}$. A superpotential is easily found by replacing in Eq. (2.9) $\pounds_{\xi}\Gamma^{\mu}_{\rho\sigma}$ by its expression in terms of $D_{\nu}\pounds_{\xi}g_{\rho\sigma}$:

$$\pounds_{\xi} \Gamma^{\mu}_{\rho\sigma} = \frac{1}{2} g^{\mu\nu} \left(D_{\rho} \pounds_{\xi} g_{\nu\sigma} + D_{\sigma} \pounds_{\xi} g_{\nu\rho} - D_{\nu} \pounds_{\xi} g_{\rho\sigma} \right). \tag{2.20}$$

The conserved current itself $\hat{I}^{\mu} = \hat{i}^{\mu} - \overline{\hat{i}^{\mu}}$ is thus also equal to the divergence of a superpotential $\hat{I}^{\mu\nu} = \hat{i}^{\mu\nu} - \overline{\hat{i}^{\mu\nu}}$. This superpotential is [see [6], on flat backgrounds see [16]; notice that flatness makes no difference in Eq. (2.21)]

$$\hat{I}^{\mu\nu} = \frac{1}{\kappa} \left(D^{[\mu} \hat{\xi}^{\nu]} - \overline{D^{[\mu} \hat{\xi}^{\nu]}} \right) + \frac{1}{\kappa} \hat{\xi}^{[\mu} k^{\nu]} = \hat{K}^{\mu\nu} + \frac{1}{\kappa} \hat{\xi}^{[\mu} k^{\nu]}, \qquad \hat{I}^{\mu} = \partial_{\nu} \hat{I}^{\mu\nu}. \tag{2.21}$$

In Eq. (2.21) $\hat{K}^{\mu\nu}$ may be called the "relative Komar superpotential", relative to the background because $(1/\kappa)D^{[\mu}\hat{\xi}^{\nu]}$, obtained with the Hilbert Lagrangian, is known as (half) the Komar [13] superpotential. $\hat{I}^{\mu\nu}$ is linear in ξ^{ν} and its first derivatives and, using Eqs. (2.4) and (2.10), can be written as follows

$$\hat{I}^{\mu\nu} = \frac{1}{\kappa} \hat{I}^{\lambda[\mu} \overline{D}_{\lambda} \xi^{\nu]} + \hat{F}^{\mu\nu}_{\lambda} \xi^{\lambda} \quad \text{with} \quad \hat{F}^{\mu\nu}_{\lambda} = \frac{1}{2\kappa} \check{g}_{\lambda\rho} \overline{D}_{\sigma} \left(\hat{g}^{\rho[\mu} \hat{g}^{\nu]\sigma} \right). \tag{2.22}$$

where $\check{g}_{\mu\nu}$ is the inverse of $\hat{g}^{\mu\nu}$. The tensor density $\hat{F}^{\mu\nu}_{\ \lambda}$ is Freud's [33] superpotential on a curved background which has been written in this form already by Cornish [34] (more on this in section 5). The great advantage of a superpotential is, as we said, to replace volume integrals of complicated vectors by surface integrals of relatively simple tensors.

(iv) How does \hat{I}^{μ} depend on a divergence in the Lagrangian density?

We found so far that Noether's method applied to the perturbed Lagrangian density, Eq. (1.1) with Eq. (1.2), generates with every smooth vector ξ^{μ} a conserved vector density \hat{I}^{μ} , Eq. (2.17), the divergence of a superpotential $\hat{I}^{\mu\nu}$, Eq. (2.21) or Eq. (2.22), and both

depend on the divergence $\partial_{\mu}\hat{k}^{\mu}$. The contribution of the divergence to our superpotential is apparent in (2.21). What is the contribution of the divergence to the different parts of the conserved vector density: the energy-momentum tensor density $\hat{\theta}^{\mu}_{\nu}$ and the helicity tensor density $\hat{\sigma}^{\mu\rho\sigma}$? To find this out let us write the divergence of the second term of Eq. (2.21) in a form similar to Eq. (2.17) with a term in ξ^{μ} and one in $\overline{D}_{\rho}\xi_{\sigma}$ and valid for any \hat{k}^{μ} :

$$\partial_{\nu} \left(\frac{1}{\kappa} \xi^{[\mu} \hat{k}^{\nu]} \right) = \frac{1}{\kappa} \overline{D}_{\lambda} \left(\delta_{\nu}^{[\mu} \hat{k}^{\lambda]} \right) \xi^{\nu} + \frac{1}{\kappa} \hat{k}^{[\rho} \overline{g}^{\mu]\sigma} \overline{D}_{\rho} \xi_{\sigma}. \tag{2.23}$$

The factors of ξ^{ν} and $\overline{D}_{\rho}\xi_{\sigma}$ represent the respective contributions in \hat{I}^{μ} to $\hat{\theta}^{\mu}_{\nu}$ and $\hat{\sigma}^{\mu\rho\sigma}$. There is no contribution to $\hat{\eta}^{\mu}$. If, in accordance with Eq. (2.1), we indicate by a prime the parts of those tensors that are independent of the divergence we may write, see Eqs. (2.17) and (2.21),

$$\hat{\theta}^{\mu}_{\nu} = \hat{\theta}'^{\mu}_{\nu} + \frac{1}{2\kappa} \left(\delta^{\mu}_{\nu} \overline{D}_{\lambda} \hat{k}^{\lambda} - \overline{D}_{\nu} \hat{k}^{\mu} \right), \tag{2.24a}$$

$$\hat{\sigma}^{\mu\rho\sigma} = \hat{\sigma}^{\prime\mu\rho\sigma} + \frac{1}{2\kappa} \left(\hat{k}^{\rho} \overline{g}^{\mu\sigma} - \hat{k}^{\mu} \overline{g}^{\rho\sigma} \right). \tag{2.24b}$$

A summary of the results obtained in this section together with comments has already been given in subsection (i).

3. Belinfante's method and Rosenfeld's identities

(i) Motivations and summary of results.

In this section we modify the conserved vector densities \hat{I}^{μ} obtained in Eq. (2.17) using Belinfante's [20] trick. In classical field theory on a Minkowski spacetime there is a problem with the canonical energy-momentum tensor, the equivalent of $\hat{\theta}^{\mu\nu} = \hat{\theta}^{\mu}_{\rho}\bar{g}^{\rho\nu}$ here. The tensor is divergenceless but not symmetrical and therefore in Minkowski coordinates the angular momentum tensor is not divergenceless; the total angular momentum is not conserved. The reason is that it does not take account of the spin of the field. It is this situation that Belinfante did remedy by changing the canonical energy-momentum tensor in such a way that the total energy momentum would remain unchanged and even the local density would still remain the same in the appropriate gauge. Rosenfeld [27] found the same correction independently and in arbitrary coordinates. We shall use Rosenfeld's method in the next section for a different purpose. The Belinfante correction has been applied to gravity in general relativity on a flat background in Minkowski coordinates by Papapetrou [21]. He could then calculate the total angular momentum at spatial infinity and give a physical meaning to some of the irreducible coefficients in asymptotic solutions of Einstein's equations.

Here we first apply the method to the electromagnetic field on a curved background. The quantities involved are familiar in electrodynamics and the results illustrate well the effect of Belinfante's modification. We then apply the same method to the conserved vector \hat{I}^{μ} of a perturbed gravitational field on an arbitrary background. We obtain in this way a new conserved vector density $\hat{\mathcal{I}}^{\mu}$, see Eq. (3.9), which generates a new energy tensor density \hat{T}^{μ}_{ν} , Eq. (3.10) with Eq. (3.8). There is also a new vector density $\hat{\mathcal{Z}}^{\mu}$ the analogue, in Eq. (2.17), of the sum $\hat{\sigma}^{\mu(\rho\sigma)}\bar{z}_{\rho\sigma}+\hat{\eta}^{\mu}$ which is zero for Killing displacements; it is defined in Eq. (3.10) and written explicitly in Eq. (3.25). We construct also a new superpotential $\hat{\mathcal{I}}^{\mu\nu}$ which is of great simplicity, see Eqs. (3.18) and (3.19). Parts of it are quite familiar in the weak field approximation. The new $\hat{\mathcal{I}}^{\mu\nu}$ discussed in some detail below in (iv) is remarkably linear in $\hat{l}^{\mu\nu}$. Even more remarkable is that the new conserved vector is independent of any divergence that is added to the Hilbert Lagrangian. The energy-momentum tensor given in Eq. (3.33) with Eq. (3.34) is particularly interesting and its properties are brought forward by Rosenfeld's identities (3.27) and (3.28). Identity (3.28) together with Eq. (3.22) shows clearly that the energy-momentum tensor density $\hat{\mathcal{T}}^{\mu\nu}$ is symmetrical only on Einstein space backgrounds defined by Eq. (3.31). Identity (3.27) shows that on such backgrounds the energy-momentum tensor density is also divergenceless.

(ii) Belinfante's correction in electro-magnetism.

The familiar example of an electromagnetic field in empty space will help to see what Belinfante's addition does to the canonical energy tensor. Let $\sqrt{-\bar{g}}\mathcal{L}^{\dagger}$ be the Lagrangian density on a flat background for simplicity. Thus $\overline{R}_{\nu\rho\sigma}^{\lambda}=0$ and in arbitrary coordinates:

$$\sqrt{-\bar{g}}\mathcal{L}^{\dagger} = -\frac{1}{16\pi}\sqrt{-\bar{g}}\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}, \qquad F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}. \tag{3.1}$$

We now repeat on $\sqrt{-\bar{g}}\mathcal{L}^{\dagger}$ the operations performed on $\hat{\mathcal{L}}$ from Eq. (2.7*a*) till Eq. (2.19). However, to avoid complications inherent to arbitrary ξ^{μ} 's, we use only Killing vectors of the background for which $\bar{z}_{\rho\sigma}=0$ but we stick to arbitrary coordinates. Then, in (daggered) notations similar to those of last section, we find that if Maxwell's equations hold $\bar{D}_{\nu}F^{\mu\nu}=0$, the conserved *vector* is of the same form as I^{μ} in Eq. (2.17) without a η^{μ} :

$$I^{\dagger \mu} = \theta^{\dagger \mu}_{\ \nu} \bar{\xi}^{\nu} + \sigma^{\dagger \mu \rho \sigma} \partial_{[\rho} \bar{\xi}_{\sigma]}, \tag{3.2}$$

Here $\sigma^{\dagger\mu\rho\sigma}$ is antisymmetric in $\rho\sigma$. The canonical energy-momentum tensor $\theta^{\dagger\mu}$ and the helicity tensor are respectively given by

$$\theta^{\dagger\mu}_{\nu} = -\frac{1}{4\pi} \left(F^{\mu\rho} \overline{D}_{\nu} A_{\rho} - \frac{1}{4} F^{\rho\sigma} F_{\rho\sigma} \delta^{\mu}_{\nu} \right), \qquad \sigma^{\dagger\mu\rho\sigma} = -\frac{1}{4\pi} F^{\mu[\rho} A^{\sigma]}. \tag{3.3}$$

The Belinfante modification consists in changing $I^{\dagger \mu}$ to

$$\mathcal{I}^{\dagger \mu} = I^{\dagger \mu} + \overline{D}_{\nu} (S^{\dagger \mu \nu \rho} \bar{\xi}_{\rho}) \tag{3.4}$$

in which

$$S^{\dagger\mu\nu\rho} = -S^{\dagger\nu\mu\rho} = \sigma^{\dagger\rho[\mu\nu]} + \sigma^{\dagger\mu[\rho\nu]} - \sigma^{\dagger\nu[\rho\mu]} = \frac{1}{4\pi}F^{\mu\nu}A^{\rho}. \tag{3.5}$$

The modified current $\mathcal{I}^{\dagger \mu}$ is now of the form

$$\mathcal{I}^{\dagger \mu} = \mathcal{T}^{\dagger \mu \nu} \bar{\xi}_{\nu}, \qquad \mathcal{T}^{\dagger \mu \nu} = \frac{1}{4\pi} \left(F^{\mu \rho} F_{\rho}^{\ \nu} + \frac{1}{4} \bar{g}^{\mu \nu} F^{\rho \sigma} F_{\rho \sigma} \right). \tag{3.6}$$

This is the familiar symmetrical, divergenceless, electro-magnetic field energy-momentum tensor.

(iii) Belinfante's correction for the conserved vectors in general relativity.

We now define, by analogy with Eq. (3.4), a new conserved vector density $\hat{\mathcal{I}}^{\mu}$ by adding to \hat{I}^{μ} , see Eq. (2.17), a divergence of an anti-symmetric tensor density constructed with the anti-symmetric part $\hat{\sigma}^{\mu[\rho\sigma]}$ of $\hat{\sigma}^{\mu\rho\sigma}$ obtained from Eq. (2.14):

$$\hat{\mathcal{I}}^{\mu} = \hat{I}^{\mu} + \partial_{\nu} \left(\hat{S}^{\mu\nu\rho} \xi_{\rho} \right) = \partial_{\nu} \left(\hat{I}^{\mu\nu} + \hat{S}^{\mu\nu\rho} \xi_{\rho} \right) = \partial_{\nu} \hat{\mathcal{I}}^{\mu\nu} \tag{3.7}$$

with

$$\hat{S}^{\mu\nu\rho} = -\hat{S}^{\nu\mu\rho} = \hat{\sigma}^{\rho[\mu\nu]} + \hat{\sigma}^{\mu[\rho\nu]} - \hat{\sigma}^{\nu[\rho\mu]}.$$
 (3.8)

The divergence added to \hat{I}^{μ} is the Belinfante addition in arbitrary coordinates. The vector density $\hat{\mathcal{I}}^{\mu}$ is linear in ξ^{μ} , in $\bar{z}_{\rho\sigma}$, see Eq. (2.11), and its derivatives $\overline{D}_{\lambda}\bar{z}_{\rho\sigma}$; $\hat{\mathcal{I}}^{\mu}$ has no term in $\partial_{[\rho}\xi_{\sigma]}$ anymore. The new conserved current is thus of the form

$$\hat{\mathcal{I}}^{\mu} = \hat{\mathcal{T}}^{\mu}_{\nu} \xi^{\nu} + \hat{\mathcal{Z}}^{\mu} = \partial_{\nu} \hat{\mathcal{I}}^{\mu\nu} \tag{3.9}$$

with

$$\hat{T}^{\mu}_{\nu} = \hat{\theta}^{\mu}_{\nu} + \bar{D}_{\rho} \hat{S}^{\mu\rho}_{\nu}, \qquad \hat{\mathcal{Z}}^{\mu} = (\hat{\sigma}^{\mu\rho\sigma} + \hat{S}^{\mu\rho\sigma}) \bar{z}_{\rho\sigma} + \hat{\eta}^{\mu}$$
 (3.10)

while

$$\hat{\mathcal{I}}^{\mu\nu} = \hat{I}^{\mu\nu} + \hat{S}^{\mu\nu\rho}\xi_{\rho}.\tag{3.11}$$

It can be seen that if ξ^{μ} is a Killing vector of the background $\bar{\xi}^{\mu}$ for which $\bar{z}_{\rho\sigma} = \hat{\eta}^{\mu} = 0$, the conserved vector is simply given by

$$\hat{\mathcal{J}}^{\mu} = \hat{\mathcal{T}}^{\mu}_{\nu} \bar{\xi}^{\nu} = \partial_{\nu} \hat{\mathcal{J}}^{\mu\nu}. \tag{3.12}$$

where $\hat{\mathcal{J}}^{\mu} = \hat{\mathcal{I}}^{\mu}(\xi = \bar{\xi})$ and $\hat{\mathcal{J}}^{\mu\nu} = \hat{\mathcal{I}}^{\mu\nu}(\xi = \bar{\xi})$.

Now consider for a moment the tensor densities $\hat{\sigma}^{\mu[\rho\sigma]}$ which makes up $\hat{S}^{\mu\nu\rho}$. The contribution from any $any \ \hat{k}^{\mu}$ to the σ -tensors have been singled out in Eq. (2.24b) from which follows, in particular, that

$$\hat{\sigma}^{\mu[\rho\sigma]} = \hat{\sigma}^{\prime\mu[\rho\sigma]} - \frac{1}{2\kappa} \bar{g}^{\mu[\rho} \hat{k}^{\sigma]}. \tag{3.13}$$

With this result we can rewrite $\hat{S}^{\mu\nu\rho}$ as a sum of a k-independent part $\hat{S}'^{\mu\nu\rho}$ and a k contribution and we find that

$$\hat{S}^{\mu\nu}_{\ \rho}\xi^{\rho} = \hat{S}'^{\mu\nu}_{\ \rho}\xi^{\rho} - \frac{1}{\kappa}\xi^{[\mu}\hat{k}^{\nu]}, \qquad \hat{S}'^{\mu\nu\rho} = \hat{\sigma}'^{\rho[\mu\nu]} + \hat{\sigma}'^{\mu[\rho\nu]} - \hat{\sigma}'^{\nu[\rho\mu]}. \tag{3.14}$$

Notice that $\hat{I}^{\mu\nu}$ in Eq. (2.21) contains exactly the same ξk -term as $\hat{S}^{\mu\nu}_{\ \rho}\xi^{\rho}$ but with the opposite sign. Thus the new superpotential (3.11) is unchanged by $\partial_{\mu}\hat{k}^{\mu}$ or any other divergence added to $\hat{\mathcal{L}}_{\underline{G}}$ for that matter. It depends only on the Hilbert Lagrangian for the perturbations $\hat{R} - \hat{R}$ and the method used to generate it. If we take Eqs. (2.21) and (3.14) into account, we may, instead of Eq. (3.11), write

$$\hat{\mathcal{I}}^{\mu\nu} = \hat{K}^{\mu\nu} + \hat{S}^{\prime\mu\nu\rho}\xi_{\rho}. \tag{3.15}$$

The divergence of this superpotential is equal to the new conserved vector $\hat{\mathcal{I}}^{\mu}$. Consequently $\hat{\mathcal{T}}^{\mu}_{\nu}$ and $\hat{\mathcal{Z}}^{\mu}$, are independent of any divergence added to the Lagrangian, contrary

to the canonical energy tensor obtained by Noether's method alone. A direct calculation, without using superpotentials gives of course the same results. Calculations are only more cumbersome.

We shall now work out explicitly the formulas for $\hat{\mathcal{I}}^{\mu\nu}$ and for $\hat{\mathcal{Z}}^{\mu}$. The explicit form of $\hat{\mathcal{T}}^{\mu}_{\nu}$ is obtained below with Rosenfeld's identities.

(iv) The explicit form of the superpotential $\hat{\mathcal{I}}^{\mu\nu}$.

The tensor $\hat{\sigma}'^{\mu[\rho\sigma]}$ is obtained from Eq. (3.13) in terms of Δ 's using Eq. (2.14) for $\hat{\sigma}^{\mu[\rho\sigma]}$ and Eq. (2.4) for \hat{k}^{μ} . The sum of the σ' 's that make up $\kappa \hat{S}'^{\mu\nu\sigma}$ in Eq. (3.14) is found to give

$$\kappa \hat{S}^{\prime\mu\nu\sigma} = \left(\Delta^{\rho}_{\rho\lambda}\hat{g}^{\sigma[\mu} - \Delta^{\sigma}_{\rho\lambda}\hat{g}^{\rho[\mu}\right)\bar{g}^{\nu]\lambda} + \left(\hat{g}^{\sigma\rho}\bar{g}^{\lambda[\mu} - \bar{g}^{\sigma\rho}\hat{g}^{\lambda[\mu}\right)\Delta^{\nu]}_{\rho\lambda} + \hat{g}^{\rho\lambda}\Delta^{[\mu}_{\rho\lambda}\bar{g}^{\nu]\sigma}. \tag{3.16}$$

On the other hand $\kappa \hat{K}^{\mu\nu}$ defined in Eq. (2.21) can be written with the help of Eq. (2.10) in this form

$$\kappa \hat{K}^{\mu\nu} = \hat{l}^{\rho[\mu} \overline{D}_{\rho} \xi^{\nu]} + \hat{g}^{\rho[\mu} \Delta^{\nu]}_{\rho\sigma} \xi^{\sigma}. \tag{3.17}$$

The sum of Eq. (3.16) times ξ_{σ} and Eq. (3.17) gives the new superpotential in terms of Δ 's. However, $\hat{\mathcal{I}}^{\mu\nu}$ has a much nicer form in terms of \overline{D} derivatives of the metric $g_{\mu\nu}$ or rather in terms of $\hat{l}^{\mu\nu}$. With Eq. (2.3) we can write $\hat{\mathcal{I}}^{\mu\nu}$ in the following form

$$\hat{\mathcal{I}}^{\mu\nu} = \frac{1}{\kappa} \hat{l}^{\rho[\mu} \overline{D}_{\rho} \xi^{\nu]} + \hat{\mathcal{P}}^{\mu\nu}{}_{\lambda} \xi^{\lambda}$$
 (3.18a)

where the \mathcal{P} -tensor plays now the role of the F tensor in (2.22):

$$\hat{\mathcal{P}}^{\mu\nu\rho} = -\mathcal{P}^{\nu\mu\rho} = \frac{1}{2\kappa} \overline{D}_{\sigma} \left(\bar{g}^{\rho\mu} \hat{l}^{\nu\sigma} - \bar{g}^{\rho\nu} \hat{l}^{\mu\sigma} - \bar{g}^{\sigma\mu} \hat{l}^{\nu\rho} + \bar{g}^{\sigma\nu} \hat{l}^{\mu\rho} \right). \tag{3.18b}$$

Another telling and useful form of the superpotential is

$$\hat{\mathcal{I}}^{\mu\nu} = \frac{1}{\kappa} \left(\hat{l}^{\rho[\mu} \overline{D}_{\rho} \xi^{\nu]} + \xi^{[\mu} \overline{D}_{\sigma} \hat{l}^{\nu]\sigma} - \bar{D}^{[\mu} \hat{l}^{\nu]}_{\sigma} \xi^{\sigma} \right). \tag{3.19}$$

The superpotential (3.19) linear in $\hat{l}^{\mu\nu}$ and its derivatives is (not surprisingly) reminiscent of various familiar expressions used in the literature and which are based on linear approximations or expansions to higher orders:

(a) On a flat background:

In Minkowski coordinates X^{μ} , $\bar{g}_{\mu\nu} = \eta_{\mu\nu} = diag(1, -1, -1, -1)$. There are ten Killing vectors in the background. The four components of the Killing vectors of translations can be taken equal to $\bar{\xi}^{\mu} = \delta^{\mu}_{\alpha}$ with $\alpha = (0, 1, 2, 3)$ and $\bar{D}_{\nu}\bar{\xi}^{\mu} = 0$; thus the four corresponding superpotentials are

$$\hat{\mathcal{P}}^{\mu\nu\alpha} = \frac{1}{2\kappa} \partial_{\sigma} \left(\eta^{\mu\alpha} \hat{g}^{\nu\sigma} + \eta^{\sigma\nu} \hat{g}^{\mu\alpha} - \eta^{\alpha\nu} \hat{g}^{\mu\sigma} - \eta^{\sigma\mu} \hat{g}^{\nu\alpha} \right). \tag{3.20}$$

This anti-symmetric tensor density $\hat{\mathcal{P}}^{\mu\nu\alpha}$ is sometimes quoted as the Papapetrou superpotential [21]. Considered as a linearized approximation $\hat{\mathcal{P}}^{\mu\nu\alpha}$ is the same quantity as Weinberg's [35] $\hat{Q}^{\mu\nu\alpha}$ and Misner, Thorne and Wheeler's [36] $\partial_{\beta}\hat{H}^{\mu\alpha\nu\beta}$. Equation (3.20) is identical with the linearized approximations of Freud's superpotential, Eq. (2.22), or of the Landau and Lifshitz superpotential written in arbitrary coordinates like in Cornish [34].

With the three spatial components for the Killing vectors of rotation which in X^{μ} coordinates are given by $\bar{\xi}^{\mu} = (\delta^{\mu}_{\alpha}\eta_{\beta\gamma} - \delta^{\mu}_{\beta}\eta_{\alpha\gamma})X^{\gamma}$, $\hat{\mathcal{I}}^{\mu\nu}$ becomes the Papapetrou superpotential for angular momentum as given in his original paper.

(b) On Einstein space backgrounds:

The linear approximation of Eq. (3.19) in arbitrary coordinates with Killing vectors $\bar{\xi}^{\mu}$ is equal to the Abbott and Deser [37] superpotential worked out on an Einstein space background [Eq. (3.31) below]. To obtain their full non-linear expression replace $\hat{l}^{\mu\nu}$ by $-\sqrt{-\bar{g}}H^{\mu\nu}$ where $H^{\mu\nu}=\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}H_{\rho\sigma}$ with $H_{\rho\sigma}$ defined in Eq. (A.13).

(c) It is important to note that the correction to $\hat{I}^{\mu\nu}$ namely $\hat{S}^{\mu\nu\rho}\xi_{\rho}$ is homogeneous of order two in $\hat{l}^{\mu\nu}$ and its derivatives. This is easily seen with Eq. (2.14) because

$$2\kappa\hat{\sigma}^{\mu[\rho\sigma]} = \left(\hat{l}^{\mu[\rho}\overline{g}^{\sigma]\nu} - \overline{g}^{\mu[\rho}\hat{l}^{\sigma]\nu}\right)\Delta^{\lambda}_{\nu\lambda} - \left(\hat{l}^{\nu[\rho}\overline{g}^{\sigma]\lambda} - \overline{g}^{\nu[\rho}\hat{l}^{\sigma]\lambda}\right)\Delta^{\mu}_{\nu\lambda}. \tag{3.21}$$

Thus $\hat{I}^{\mu\nu}$ and $\hat{\mathcal{I}}^{\mu\nu}$ are equal in the linear approximation as well on arbitrary backgrounds. From this follows [see KBL97] that $\hat{\mathcal{I}}^{\mu\nu}$ provides the correct energy and linear momentum at spatial infinity. It gives also correctly the Bondi [15] - Sachs [38] energy and linear momentum at null infinity (see [39]). Details and proofs are given in appendix where we show also that the Abbott and Deser superpotential does not give the Bondi-Sachs linear momentum.

(v) The new $\hat{\mathcal{Z}}^{\mu}$ vector density.

This vector density plays a role in conservation laws that are not associated with Killing vectors as in the examples mentioned in the introduction and in the cosmological applications of section 4. The vector density has also an important task in Rosenfeld's identities (see below). \hat{Z}^{μ} is defined in Eq. (3.10). It must be noted in Eq. (3.9) that the factor of $\overline{D}_{\rho}\xi_{\sigma}$ is indeed symmetrical in $\rho\sigma$: This follows from the definition (3.8) of $\hat{S}^{\mu\sigma\rho}$. We need a new symbol for that factor which plays a role later on; let us set

$$*\hat{S}^{\mu\rho\sigma} = *\hat{S}^{\mu\sigma\rho} = \hat{\sigma}^{\mu\sigma\rho} + \hat{S}^{\mu\sigma\rho} = \hat{\sigma}^{(\mu\rho)\sigma} + \hat{\sigma}^{(\mu\sigma)\rho} - \hat{\sigma}^{(\rho\sigma)\mu} = \hat{\sigma}'^{(\mu\rho)\sigma} + \hat{\sigma}'^{(\mu\sigma)\rho} - \hat{\sigma}'^{(\rho\sigma)\mu}.$$
(3.22)

Thus

$$\hat{\mathcal{Z}}^{\mu} = *\hat{S}^{\mu\rho\sigma}\bar{z}_{\rho\sigma} + \hat{\eta}^{\mu} \tag{3.23}$$

with $\hat{\eta}^{\mu}$ defined by Eq. (2.19). An interesting form of $*\hat{S}^{\mu\rho\sigma}$ comes from using Eq. (2.14) for $\hat{\sigma}^{(\mu\rho)\sigma}$:

$$*\hat{S}^{\mu\rho\sigma} = \frac{1}{2\kappa} \overline{D}_{\nu} \left(2\hat{l}^{\mu(\rho} \bar{g}^{\sigma)\nu} - \bar{g}^{\mu\nu} \hat{l}^{\rho\sigma} - \hat{l}^{\mu\nu} \bar{g}^{\rho\sigma} \right). \tag{3.24}$$

Plugging Eqs. (3.24) and (2.19) into $\hat{\mathcal{Z}}^{\mu}$ gives the following form "anti-symmetric" in $\bar{z} \leftrightarrow l$

$$2\kappa\hat{\mathcal{Z}}^{\mu} = 2\left(\bar{z}^{\rho\sigma}\overline{D}_{\rho}\hat{l}^{\mu}_{\sigma} - \hat{l}^{\rho\sigma}\overline{D}_{\rho}\bar{z}^{\mu}_{\sigma}\right) - \left(\bar{z}_{\rho\sigma}\overline{D}^{\mu}\hat{l}^{\rho\sigma} - \hat{l}^{\rho\sigma}\overline{D}^{\mu}\bar{z}_{\rho\sigma}\right) + \left(\hat{l}^{\mu\nu}\overline{D}_{\nu}\bar{z} - \bar{z}\overline{D}_{\nu}\hat{l}^{\mu\nu}\right). \tag{3.25}$$

(v) The Rosenfeld identities.

The conservation law $\partial_{\mu}\hat{\mathcal{I}}^{\mu}=0$ which holds for any smooth vector ξ^{μ} contains derivatives of ξ^{μ} of an order as high as 3. Thus with the help of Eq. (3.9), $\partial_{\mu}\hat{\mathcal{I}}^{\mu}=0$ can be written in the form

$$\partial_{\mu}\hat{\mathcal{I}}^{\mu} = \partial_{\mu}(\hat{\mathcal{T}}^{\mu}_{\nu}\xi^{\nu}) + \partial_{\mu}\hat{\mathcal{Z}}^{\mu} = \hat{\beta}_{\nu}\xi^{\nu} + \hat{\beta}^{\mu}_{\nu}\overline{D}_{\mu}\xi^{\nu} + \hat{\beta}^{\rho\sigma}_{\nu}\overline{D}_{(\rho\sigma)}\xi^{\nu} + \hat{\beta}^{\mu\rho\sigma}_{\nu}\overline{D}_{(\mu\rho\sigma)}\xi^{\nu} = 0. \quad (3.26)$$

This identity holds for arbitrary smooth ξ 's. Therefore all the properly symmetrized β 's must be identically zero, $\hat{\beta}_{\nu} = \hat{\beta}^{\mu}_{\ \nu} = \hat{\beta}^{(\rho\sigma)}_{\ \nu} = \hat{\beta}^{(\mu\rho\sigma)}_{\ \nu} = 0$. This is the way in which Rosenfeld [27] obtained a set of identities and found the Belinfante correction. \hat{T}^{μ}_{ν} appears obviously in $\hat{\beta}^{\mu}_{\ \nu} = 0$ and $\overline{D}_{\mu}\hat{T}^{\mu}_{\nu}$ in $\hat{\beta}_{\nu} = 0$. The remaining identities are independent of the energy tensor. In Rosenfeld's work, which is in arbitrary coordinates but on a flat background, $\hat{\beta}^{\mu}_{\ \nu} = 0$ was the most interesting relation. It related the canonical energy momentum generated by Noether's method to the symmetric and divergenceless energy-momentum needed in Einstein's equations. The divergencelessness was shown by $\hat{\beta}_{\nu} = 0$.

Here, however, the background is not flat and Rosenfeld's identities give different and interesting results. It is obvious that the calculations of the β 's asks for a lot of rearrangements of factors that come exclusively from $\partial_{\mu}\hat{\mathcal{Z}}^{\mu}$. This is somewhat tedious but really straightforward. The resulting identities have in the end a nice form:

$$\hat{\beta}_{\nu} = \overline{D}_{\mu}\hat{\mathcal{T}}^{\mu}_{\nu} - \frac{1}{2\kappa}\hat{l}^{\rho\sigma}\overline{D}_{\mu}\overline{R}_{\rho\sigma} = 0, \tag{3.27}$$

$$\hat{\beta}^{\mu\nu} = \hat{\beta}^{\mu}_{\ \rho} \bar{g}^{\rho\nu} = \hat{T}^{\mu\nu} + \overline{D}_{\rho} (*\hat{S}^{\rho\mu\nu}) - \frac{1}{\kappa} \hat{l}^{\rho\mu} \bar{R}^{\nu}_{\rho} = 0, \tag{3.28}$$

$$\hat{\beta}^{(\rho\sigma)\nu} = *\hat{S}^{(\rho\sigma)\nu} + \overline{D}_{\mu}\hat{\beta}^{\mu\rho\sigma\nu} = 0, \tag{3.29}$$

$$\hat{\beta}^{(\mu\rho\sigma)\nu} = 0 \quad \text{with} \quad \hat{\beta}^{\mu\rho\sigma\nu} = \hat{\beta}^{\mu\sigma\rho\nu} = \frac{1}{2\kappa} \left(\hat{l}^{\mu(\rho} \bar{g}^{\sigma)\nu} - \hat{l}^{\rho\sigma} \bar{g}^{\mu\nu} \right). \tag{3.30}$$

If we remember that $*\hat{S}^{\rho\mu\nu}$ is symmetrical in $\mu\nu$, see Eq. (3.24), we see from Eq. (3.28) that $\hat{T}^{\mu\nu} = \hat{T}^{\nu\mu}$ if and only if

$$\hat{l}^{\rho[\mu}\bar{R}^{\nu]}_{\rho} = 0 \quad \to \quad \overline{R}_{\mu\nu} = -\overline{\Lambda}\bar{g}_{\mu\nu} \tag{3.31}$$

where $\overline{\Lambda}$ is necessarily a constant. Thus the energy-momentum tensor is symmetrical on backgrounds that are *Einstein spaces* in the sense of A.Z. Petrov [24]. de Sitter and Einstein's cosmological *spacetimes* belong to this class but in general Friedmann-Robertson-Walker spacetimes do not. This is why our formalism with arbitrary ξ^{μ} 's is precisely good in relativistic cosmology and why \hat{Z}^{μ} may be important in that case.

When the background is an Einstein space, Eq. (3.27) shows that $\hat{T}^{\mu\nu}$ is also divergenceless. Identity (3.28) provides an interesting form for $\hat{T}^{\mu\nu}$; using Eq. (3.24) in Eq. (3.28) we can write $\hat{T}^{\mu\nu}$ like this:

$$\hat{T}^{\mu\nu} = -\overline{D}_{\rho}\hat{\mathcal{P}}^{\rho\nu\mu} + \frac{3}{2\kappa}\hat{l}^{\rho[\mu}\bar{R}^{\nu]}_{\rho} + \frac{1}{2\kappa}[\hat{l}^{\rho(\mu}\bar{R}^{\nu)}_{\rho} - \hat{l}^{\rho\sigma}\bar{R}^{\mu}_{\rho\sigma}^{\nu}]. \tag{3.32}$$

On a flat background we see, looking at Eq. (3.20), that $\hat{T}^{\mu\nu}$ is the second order derivative of a tensor which was also obtained by Papapetrou [21]. Thus Eq. (3.32) is the generalization of his equation to curved backgrounds.

The new energy-momentum tensor can be calculated from Eq. (3.10) with the use of Eqs. (2.13), (2.14), (2.18) and (3.8). It contains three types of terms, very similar to those of $\hat{\theta}^{\mu\nu}$: a symmetric matter energy momentum of the perturbations, a symmetric field energy-momentum tensor $\hat{\tau}^{\mu\nu} = \hat{\tau}^{\nu\mu}$ and two non-derivative couplings to the Ricci tensor of the background, the second of which being antisymmetrical:

$$\hat{T}^{\mu\nu} = (\hat{T}^{(\mu}_{\rho} \overline{g}^{\nu)\rho} - \overline{\hat{T}^{\mu\nu}}) + \hat{\tau}^{\mu\nu} + \frac{1}{2\kappa} \hat{l}^{\rho\sigma} \overline{R}_{\rho\sigma} \overline{g}^{\mu\nu} + \frac{1}{\kappa} \hat{l}^{\lambda[\mu} \overline{R}^{\nu]}_{\lambda}. \tag{3.33}$$

The field energy-momentum tensor density is the following terrifying homogeneous quadratic form in $\hat{l}^{\mu\nu}$, their first and second order derivatives:

$$\kappa \hat{\tau}^{\mu\nu} = \frac{1}{2} \left(\hat{l}^{\mu\nu} \overline{g}^{\rho\sigma} - \overline{g}^{\mu\nu} \hat{l}^{\rho\sigma} \right) \overline{D}_{\sigma} \Delta^{\lambda}_{\rho\lambda} + \left(\hat{l}^{\rho\sigma} \overline{g}^{\lambda(\mu} - \overline{g}^{\rho\sigma} \hat{l}^{\lambda(\mu}) \right) \overline{D}_{\sigma} \Delta^{\nu)}_{\lambda\rho}
+ \overline{g}^{\rho\sigma} \left(\frac{1}{2} \hat{g}^{\mu\nu} \Delta^{\lambda}_{\rho\lambda} \Delta^{\eta}_{\sigma\eta} + \hat{g}^{\lambda\eta} \Delta^{(\mu}_{\lambda\rho} \Delta^{\nu)}_{\eta\sigma} + \Delta^{\lambda}_{\sigma\eta} \Delta^{(\mu}_{\lambda\rho} \hat{g}^{\nu)\eta} - 2\Delta^{\lambda}_{\sigma\lambda} \Delta^{(\mu}_{\eta\rho} \hat{g}^{\nu)\eta} \right)
+ \hat{g}^{\lambda\eta} \left[\frac{1}{2} \overline{g}^{\mu\nu} \Delta^{\sigma}_{\rho\lambda} \Delta^{\rho}_{\sigma\eta} + \left(\Delta^{\sigma}_{\rho\sigma} \Delta^{(\mu}_{\lambda\eta} - \Delta^{\sigma}_{\lambda\eta} \Delta^{(\mu}_{\rho\sigma} - \Delta^{\sigma}_{\lambda\rho} \Delta^{(\mu}_{\eta\sigma}) \overline{g}^{\nu)\rho} \right].$$
(3.34)

On Ricci flat backgrounds $\hat{T}^{\mu\nu}$, Eq. (3.33), reduces to the expression found by Grishchuk et al [40].

(vii) Second derivatives in the energy tensor?

Second derivatives of $g_{\mu\nu}$ appear in the field energy tensor. This needs some comments. The canonical field energy \hat{t}^{μ}_{ν} , see Eq. (2.13), is quadratic in first order derivatives. Thus that tensor density depends certainly on initial conditions only. This is the normal behavior of a conserved quantity.

Notice that the volume integral of the new conserved vector density $\hat{\mathcal{I}}^{\mu}$ is equal to a surface integral of $\hat{\mathcal{I}}^{\mu\nu}$ in which there are no more than first order derivatives. This is also a suitable result.

Consider now the local quantities $\hat{\mathcal{I}}^{\mu}$ themselves. Suppose that initial conditions are defined on a hypersurface at a given time coordinate $x^0 = 0$. Initial conditions are the metric components and their time derivatives (modulo Einstein's constraints and gauge freedom). The coordinate density is equal to $\hat{\mathcal{I}}^0 = \hat{I}^0 + \partial_k (\hat{S}^{0k\sigma}\xi_{\sigma})$ with k = 1, 2, 3. Only spatial derivatives of S appear in $\hat{\mathcal{I}}^0$ because S is anti-symmetric in the first two indices. Thus since \hat{I}^0 and $\hat{S}^{0k\sigma}\xi_{\sigma}$ contain only first order time derivatives, $\hat{\mathcal{I}}^0$ itself contains only first order time derivatives and therefore even the energy-momentum density contains no more than first order time derivatives. The Belinfante correction adds only second order spatial derivatives. It is sometimes required, like in KBL97 that the energy-momentum tensor should contain no more than first order derivatives. This requirement may be unnecessarily restrictive.

(viii) Conservation laws directly from Einstein's equations?

There is no difficulty to rebuild Einstein's equations $\hat{G}^{\mu}_{\nu} = \kappa \hat{T}^{\mu}_{\nu}$ from the conservation laws $\partial_{\nu}\hat{\mathcal{I}}^{\mu\nu} = \hat{\mathcal{I}}^{\mu}$. To do this we rewrite $\hat{\mathcal{I}}^{\mu\nu}$ given by (3.19) as follows

$$\kappa \hat{\mathcal{I}}^{\mu\nu} = -\bar{D}^{[\mu}\hat{l}^{\nu]}_{\rho}\xi^{\rho} + \xi^{[\mu}\hat{\mathcal{G}}^{\nu]} + \hat{l}^{\rho[\mu}\bar{D}_{\rho}\xi^{\nu]}$$
(3.35)

where $\hat{\mathcal{G}}^{\nu} = \bar{D}_{\rho}\hat{l}^{\rho\nu}$. It is useful to keep track of $\hat{\mathcal{G}}^{\nu}$ because $\hat{\mathcal{G}}^{\nu} = 0$ is the familiar (generalization of the) well known De Donder gauge condition. Thus, if we take the divergence of Eq. (3.35) and remember that $\partial_{\nu}\hat{\mathcal{I}}^{\mu\nu} = \hat{\mathcal{I}}^{\mu}$ we find, by replacing $\hat{\mathcal{I}}^{\mu}$ with $\hat{\mathcal{T}}^{\mu}_{\nu}\xi^{\nu} + \hat{\mathcal{Z}}^{\mu}$ and $\hat{\mathcal{T}}^{\mu\nu}$ by its expression given in Eq. (3.33) that

$$2\kappa\partial_{\nu}\hat{\mathcal{I}}^{\mu\nu} = \left(\bar{D}_{\rho}\bar{D}^{\rho}\hat{l}^{\mu\nu} - 2\bar{D}^{(\mu}\hat{\mathcal{G}}^{\nu)} + \bar{D}_{\rho}\hat{\mathcal{G}}^{\rho}\bar{g}^{\mu\nu} + 2\hat{l}^{\rho[\mu}\bar{R}^{\nu]}_{\rho} - 2\hat{l}^{\rho\sigma}\bar{R}^{\mu\nu}_{\rho\sigma}\right)\xi_{\nu} + 2\kappa\hat{\mathcal{Z}}^{\mu}$$

$$= 2\kappa\hat{\mathcal{I}}^{\mu} = \left[2\kappa\left(\hat{T}^{(\mu}_{\rho}\bar{g}^{\nu)\rho} - \overline{\hat{T}^{\mu\nu}} + \hat{\tau}^{\mu\nu}\right) + 2\hat{l}^{\rho[\mu}\bar{R}^{\nu]}_{\rho} + \hat{l}^{\rho\sigma}\bar{R}_{\rho\sigma}\bar{g}^{\mu\nu}\right]\xi_{\nu} + 2\kappa\hat{\mathcal{Z}}^{\mu}.$$
(3.36)

We can remove $2\kappa\hat{Z}^{\mu}$ and $2\hat{l}^{\rho[\mu}\bar{R}^{\nu]}_{\rho}\xi_{\nu}$ from both sides of Eq. (3.36). The remaining homogeneous linear expression in ξ_{ν} is true for any ξ_{ν} . The factors of ξ_{ν} on both sides of the equality must thus be equal. We are left with a set of equations that are of course Einstein's equations in which the left hand side contains all terms that are linear in $\hat{l}^{\mu\nu}$:

$$\bar{D}_{\rho}\bar{D}^{\rho}\hat{l}^{\mu\nu} - 2\bar{D}^{(\mu}\hat{\mathcal{G}}^{\nu)} + \bar{D}_{\rho}\hat{\mathcal{G}}^{\rho}\bar{g}^{\mu\nu} - \hat{l}^{\rho\sigma}\bar{R}_{\rho\sigma}\bar{g}^{\mu\nu} - 2\hat{l}^{\rho\sigma}\bar{R}^{\mu}_{\ \rho\sigma}{}^{\nu} = 2\kappa \left(\hat{T}^{(\mu}_{\rho}\bar{g}^{\nu)\rho} - \overline{\hat{T}^{\mu\nu}} + \hat{\tau}^{\mu\nu}\right).$$
(3.37)

On a flat background, in Minkowski coordinates these are Einstein's equations as they were written down by Papapetrou [21]. Equations (3.37) have also been given in this form by Grishchuk *et al* [40] on a Ricci flat background $\bar{R}_{\rho\sigma} = 0$.

The linearized approximation on a flat background with the De Donder gauge condition is readily recognized as the gravitational wave equations written in arbitrary coordinates:

$$\bar{D}_{\rho}\bar{D}^{\rho}(\sqrt{-g}g^{\mu\nu}) = 2\kappa\hat{T}^{\mu\nu}.$$
(3.38)

Equation (3.37) is an interesting form of Einstein's equations from which we could have constructed our conservation laws $\partial_{\nu}\hat{\mathcal{I}}^{\mu\nu} = \hat{\mathcal{I}}^{\mu}$. But who has thought that by adding $2\kappa\hat{\mathcal{Z}}^{\mu} + 2\hat{l}^{\rho[\mu}\bar{R}^{\nu]}_{\rho}\xi_{\nu}$ on both sides of Eq. (3.37) we would get $\partial_{\nu}\hat{\mathcal{I}}^{\mu\nu} = \hat{\mathcal{I}}^{\mu}$?

4. Conformal Killing vectors, conservation laws and integral constraints in cosmology

(i) Motivations and summary of results.

Here we illustrate the theory developed in the previous sections with some applications in theoretical cosmology. Conservation laws have been used previously in cosmology (see introduction) and the following examples give potentially useful new formulas.

We start by considering Friedmann-Robertson-Walker backgrounds with their 15 conformal Killing vectors. We take the metric of the backgrounds in the form given by Eq. (4.1) and with Eq. (4.2) in terms of a conformal time η . In these coordinates the conformal Killing vectors satisfy the 10 equations given in Eq. (4.6). A set of 15 linearly independent solutions of those equations is given in Eqs. (4.7) and (4.8) for spacetimes with flat spacelike sections $\eta = 0$ (k = 0) and in Eqs. (4.7) and (4.10) for curved spacelike sections ($k = \pm 1$). Conserved quantities and integral constraints at a given time η for perturbations of the background will be obtained from the time components of the conserved vector densities $\hat{\mathcal{I}}^0$ and the corresponding superpotential components $\hat{\mathcal{I}}^{0l}$ (l = 1, 2, 3). These components can be calculated for the 15 conformal Killing vectors and the general formulas for that are given by Eqs. (4.18a) or (4.18b) and (4.19). Equation (4.18a) provide an expression for perturbations that may even be large. The rest of this section deals mainly with applications of these formulas.

The 15 integrands $*\mathcal{I}$ and $*\mathcal{I}^l$ defined in Eq. (4.20) are calculated explicitly. Notice that Eqs. (4.18) and (4.19) contain components of the conformal Killing vectors and their spatial derivatives. They do not contain the time derivatives of these components. We may thus use $*\mathcal{I}$'s and $*\mathcal{I}^l$'s obtained from linear combinations of conformal Killing vectors with time dependent coefficients. The reason for doing this is that we find in this way 15 relatively simple integrands. They are written in Eqs. (4.24) for k=0 and in Eqs. (4.24) for $k=\pm 1$. Notice that linear combinations of conformal Killing vectors are not conformal Killing vectors anymore. Nevertheless with those non conformal Killing vectors we have obtained interesting "conservation laws". We show, for instance, that Traschen's integral constraint vectors [1] are equal to time dependent linear combinations of conformal Killing vectors. Formulae are explicitly given in Eq. (4.25) for k=0 and Eq.

(4.26) for $k = \pm 1$. We also show that if we equate to zero the uniform Hubble expansion rate \mathcal{Q} [26] defined in Eq. (4.17), 14 of the 15 integral constraints take the particularly simple form shown in Eq. (1.10). The lone exception is associated with time translations $(k = \pm 1)$ or time accelerations (k = 0). These exceptions are precisely those conservation laws that interested Uzan et al [8] (see also [7]) in which an unexpected field contribution in the conservation law of "energy" is inevitable. We also give a non trivial illustration of a spacetime that is asymptotically Schwarzschild-de Sitter with k = 0. In this case we obtain 13 global integral constraints with no boundary contributions and two non-zero constants of motion for perturbations that may be large near the source but weak at infinity. The results, which may be new, are given in Eq. (4.34). For perturbations that are small everywhere, the formulas are given by Eq. (4.35).

(ii) Friedmann-Robertson-Walker spacetimes and their conformal Killing vectors.

We write the background metric $d\bar{s}^2$ in dimensionless coordinates $x^{\mu}=(x^0=\eta,\ x^k)$ with k,l,m=1,2,3 for which the symmetrical role of x^k is apparent:

$$d\bar{s}^2 = \bar{g}_{\mu\nu} dx^{\mu} dx^{\nu} = a^2 (d\eta^2 - f_{kl} dx^k dx^l) = a^2 e_{\mu\nu} dx^{\mu} dx^{\nu}, \tag{4.1}$$

 $a(\eta)$ is the scale factor and f_{kl} , f^{kl} and $f = det(f_{kl})$ are respectively given by

$$f_{kl} = \delta_{kl} + k \frac{\delta_{km} x^m \delta_{ln} x^n}{1 - kr^2}, \quad f^{kl} = \delta^{kl} - k x^k x^l, \quad f = \frac{1}{1 - kr^2},$$
 (4.2)

k=0 or ± 1 and $r^2=\delta_{kl}x^kx^l$. The non-zero Christoffel symbols of the metric (4.1) are

$$\bar{\Gamma}_{00}^0 = \dot{a}, \quad \bar{\Gamma}_{kl}^0 = \dot{a}f_{kl}, \quad \bar{\Gamma}_{0l}^m = \dot{a}\delta_l^m, \quad \bar{\Gamma}_{kl}^m = kx^m f_{kl}, \tag{4.3}$$

à is the dimensionless conformal Hubble "constant"

$$\dot{a} = \frac{1}{a} \frac{da}{d\eta}.\tag{4.4}$$

In these notations the non zero components of the Einstein tensor are respectively

$$\overline{G}_{0}^{0} = \frac{3}{a^{2}}(k + \dot{a}^{2}) = \kappa \overline{T}_{0}^{0}, \quad \overline{G}_{l}^{m} = \frac{1}{a^{2}}(k + \dot{a}^{2} + 2\partial_{0}\dot{a})\delta_{l}^{m} = \kappa \overline{T}_{l}^{m}, \tag{4.5}$$

The conformal Killing vectors are the 15 linearly independent solutions $\xi^{\mu} = (\xi^0, \xi^k)$ of Eq. (1.8). These equations are *independent* of the conformal factor and can be written in 3-dimensional notations as follows:

$$\partial_0 \xi^0 = \frac{1}{3} \nabla_k \xi^k, \qquad \partial_0 \xi^k = \nabla^k \xi^0, \qquad \nabla^{(k} \xi^{l)} = f^{kl} \partial_0 \xi^0. \tag{4.6}$$

where ∇_k is a 3-covariant derivative for the f_{kl} metric, $\nabla^k = f^{kl}\nabla_l$, and the first equation equals one third of the trace of the last one. We found the solutions of Eq. (4.6) as follows. (a) The group of conformal transformations in Minkowski coordinates X^{μ} is explicitly given in [25]; infinitesimal transformations provide the conformal Killing vectors of the flat spacetime in X^{μ} coordinates. (b) The metric $e_{\mu\nu}$ is conformal to the Minkowski metric $\eta_{\mu\nu}$, i.e. in appropriate coordinates $e_{\mu\nu} = \Omega^2 \eta_{\mu\nu}$ globally and the conformal Killing vector components are the same since they do not dependent on Ω ; we had thus only to take the components of ξ^{μ} in Minkowski coordinates and transform them into our coordinates x^{μ} ; this is easily calculated from the explicit global coordinate transformations given in [41]. (c) One can in the end verify that the results satisfy indeed Eq. (4.6). Here are the results.

There are 7 conformal Killing vectors which can be written in compact form for every value of k; these are the conformal Killing vectors of time accelerations (\mathbf{t}), space translations (\mathbf{s}_a , a = 1, 2, 3) and space rotations (\mathbf{r}_a):

$$t^{\mu} = \delta_0^{\mu}, \quad s_a^{\mu} = \delta_a^{\mu} \sqrt{1 - kr^2}, \quad r_a^{\mu} = \delta^{\mu k} \epsilon_{kal} x^l.$$
 (4.7)

The other 8 conformal Killing vectors are somewhat different for k=0 and for $k=\pm 1$.

For k = 0, the Lorentz boosts (\mathbf{l}_a), dilatation (\mathbf{d}), time acceleration (\mathbf{a}_0) and space accelerations (\mathbf{a}_a) (the last two have been studied in [25]) are respectively given by

$$\mathbf{l}_{a} = (l_{a}^{0} = x^{a}, \ l_{a}^{k} = \eta \delta_{a}^{k}), \quad \mathbf{d} = (d^{0} = \eta, \ d^{k} = x^{k}),$$

$$\mathbf{a}_{0} = (a_{0}^{0} = \eta^{2} + r^{2}, \ a_{0}^{k} = 2\eta x^{k}), \quad \mathbf{a}_{a} = (a_{a}^{0} = 2\eta x^{a}, \ a_{a}^{k} = 2x^{k} x^{a} + [\eta^{2} - r^{2}]\delta_{a}^{k}). \quad (4.8)$$

$$(k = 0)$$

For $k=\pm 1$ the 8 vectors can be written in a more compact form in terms of the column matrix

$$\beta = \begin{pmatrix} \beta^{\bullet} \\ \beta_{\bullet} \end{pmatrix} = \begin{pmatrix} \alpha \\ \partial_0 \alpha \end{pmatrix} \text{ with } \alpha = \sin \eta \ (k = 1) \text{ or } \alpha = \sinh \eta \ (k = -1). \tag{4.9}$$

What in flat spacetime corresponds to dilatation and time acceleration can be written as a single combination (δ); the same is true of what correspond to the 3 Lorentz boosts and 3 space accelerations (λ_a)

$$\delta = \begin{pmatrix} \delta^{\bullet} \\ \delta_{\bullet} \end{pmatrix} = (\delta^{0} = \beta \sqrt{1 - kr^{2}}, \ \delta^{k} = \partial_{0}\beta \sqrt{1 - kr^{2}}x^{k}),
\lambda_{a} = \begin{pmatrix} \lambda^{\bullet} \\ \lambda_{\bullet} \end{pmatrix} = (\lambda_{a}^{0} = \partial_{0}\beta x^{a}, \ \lambda_{a}^{k} = \beta f^{ka}).$$

$$(k = \pm 1) \qquad (4.10)$$

Notice that for k = 0 we can take $\alpha = \eta$ and apply Eq. (4.10) to that case as well. Then $\beta = \binom{\eta}{1}$ and $\delta = \binom{\mathbf{d}}{\mathbf{t}}$ while $\lambda_a = \binom{\mathbf{l}_a}{\mathbf{s}_a}$. These 4 vectors are not the same as those given in Eq. (4.8); only two of them are in that group \mathbf{d} and \mathbf{l}_a , the other two \mathbf{t} and \mathbf{s}_a belong to the group of Eq. (4.7).

15 conformal Killing vectors are given by any linear combination (that are linearly independent of course) with constant coefficients of Eqs. (4.7) and (4.8) for k = 0, Eqs. (4.7) and (4.10) for $k = \pm 1$.

Conformal Killing vectors and their space derivatives appear in the zero component of the conserved vector densities $\hat{\mathcal{I}}^0$ [see Eq. (4.19) below] in two combinations $\frac{1}{4}\bar{z}$ and a new one that we denote by \bar{y} :

$$\frac{1}{4}\bar{z} = \frac{1}{4}\overline{D}_{\rho}\xi^{\rho} = \frac{1}{3}\nabla_{k}\xi^{k} + \dot{a}\xi^{0}, \quad \bar{y} \equiv (k + \dot{a}^{2} - \partial_{0}\dot{a})\xi^{0} + \frac{1}{4}(\partial_{0}\bar{z} - \dot{a}\bar{z}) = \frac{1}{3}\nabla^{2}\xi^{0} + k\xi^{0}. \quad (4.11)$$

Most of the \bar{y} 's are zero. The non-zero one's are $\bar{y}(\mathbf{t}) = k$ and $\bar{y}(\mathbf{a}_0) = 2$.

There are 3-antisymmetric tensors $\nabla^{[k}\xi^{l]}$ which appear in the \mathcal{I}^{0l} components of the superpotential, see Eq. (4.18*a*) or Eq. (4.18*b*). The tensors are not zero for the following three conformal Killing vectors,

$$\nabla^{[k} s_a^{l]} = -2kx^{[k} s_a^{l]}, \quad \nabla^{[k} r_a^{l]} = \epsilon^{akl} - 2kx^{[k} r_a^{l]}, \quad (k = 0, \pm 1)$$

$$\nabla^{[k} a_a^{l]} = 4\delta_a^{[k} x^{l]}. \qquad (k = 0)$$
(4.12)

(iii) Superpotentials and conserved vectors for small perturbations.

We denote the perturbed metric components $g_{\mu\nu}$ by $\bar{g}_{\mu\nu} + h_{\mu\nu}$. Some authors [42] prefer to use the "conformal perturbations" and write $g_{\mu\nu} = a^2(e_{\mu\nu} + \tilde{h}_{\mu\nu})$. Thus,

$$ds^{2} = (\bar{g}_{\mu\nu} + h_{\mu\nu})dx^{\mu}dx^{\nu} = a^{2}(e_{\mu\nu} + \tilde{h}_{\mu\nu})dx^{\mu}dx^{\nu}. \tag{4.13}$$

We shall mainly use $\tilde{h}_{\mu\nu}$; $h_{\mu\nu}$ seems to be preferable in 4-covariant perturbation calculations. In a 1+3 splitting, the 10 components of the perturbations are \tilde{h}_{00} , \tilde{h}_{0l} , \tilde{h}_{kl} . We shall not displace the 0-indices up or down. The kl indices will be displaced with the f_{kl} metric. Thus

$$\tilde{h}_{l}^{m} = f^{mk}\tilde{h}_{kl}, \quad \tilde{h}^{mn} = f^{mk}f^{nl}\tilde{h}_{kl}, \quad \tilde{h}_{0}^{m} = f^{ml}\tilde{h}_{0l}.$$
 (4.14)

There are simple relations between h^{μ}_{ν} and $\tilde{h}_{\mu\nu}$: $h^0_0 = \tilde{h}_{00}$, $h^0_l = \tilde{h}_{0l}$ and $h^k_l = -\tilde{h}^k_l$. The tensor density that enters in the superpotential (3.18) is related to $h_{\mu\nu}$ in the linear approximation as follows

$$\hat{l}^{\mu\nu} = \sqrt{-\bar{g}}(-\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma} + \frac{1}{2}\bar{g}^{\mu\nu}\bar{g}^{\rho\sigma})h_{\rho\sigma} = \sqrt{f}(-e^{\mu\rho}e^{\nu\sigma} + \frac{1}{2}e^{\mu\nu}e^{\rho\sigma})h_{\rho\sigma}.$$
 (4.15)

Notice that Eq. (3.18) in terms of $\hat{l}^{\mu\nu}$ is the same for large or small perturbations and can be used for calculations at higher order of approximations. In this section, we are interested

only in small perturbations. There are two 3-tensors that deserve special notations because they appear as building blocks of the components of $\hat{\mathcal{I}}^{0l}$:

$$q_l^m \equiv \delta_l^m \tilde{l}^{00} - \tilde{l}_l^m, \quad Q_l^m \equiv \nabla^m \tilde{l}_l^0 - \partial_0 q_l^m - \delta_l^m \left[\nabla^n \tilde{l}^{0n} + \dot{a} (\tilde{l}_n^n + \tilde{l}^{00}) \right], \quad \tilde{l}^{\mu\nu} \equiv (a^2 \sqrt{-\bar{g}}) \hat{l}^{\mu\nu}.$$
(4.16a)

In the linear approximation where Eq. (4.15) holds the quantities in Eq. (4.16a) reduce to

$$q_l^m = \tilde{h}_l^m - \delta_l^m \tilde{h}_n^n, \quad Q_l^m = (2\dot{a}\tilde{h}_{00} - \nabla^n \tilde{h}_{0n})\delta_l^m + \nabla^m \tilde{h}_{0l} - \partial_0 q_l^m. \tag{4.16b}$$

We also define by a special symbol Q the perturbed trace of the external curvature of the hypersurface $\eta = const$ which appear in the zero component of the conserved vectors. Thus, if n^{μ} is the unit normal vector to that hypersurface,

$$Q \equiv -D_{\mu}n^{\mu} - (-\overline{D_{\mu}n^{\mu}}) = \frac{3}{2}\dot{a}\tilde{h}_{00} + \frac{1}{2}\partial_{0}\tilde{h}_{n}^{n} - \nabla_{n}\tilde{h}_{0}^{n}. \tag{4.17}$$

Q = 0 is the "uniform Hubble expansion" gauge condition which was introduced by Bardeen [26].

We have now all the elements needed to calculate the conserved vectors and superpotentials with small perturbations for the 15 conformal Killing vectors and to write them down in a compact form. We are particularly interested in integral constraints over volumes at a constant time with spherical boundaries. For this we need only the zero components of the conserved vector: $\hat{\mathcal{I}}^0 = \partial_l \hat{\mathcal{I}}^{0l}$. Let us write first $\hat{\mathcal{I}}^{0l}$ which is define by Eq. (3.18). We obtain after painful but straightforward calculations the following expression for the superpotential components which are valid for large perturbations, using Eq. (4.16a)

$$\hat{\mathcal{I}}^{0l} = \frac{\sqrt{-\bar{g}}}{2\kappa a^2} \left[(2\dot{a}\tilde{l}^{0l} - \nabla^k q_k^l)\xi^0 + q_k^l \nabla^k \xi^0 + Q_k^l \xi^k + \tilde{l}_k^0 \nabla^{[k} \xi^{l]} \right]. \tag{4.18a}$$

For small perturbations $h_{\mu\nu}$ Eq. (4.18a) reduces with the help of Eq. (4.15) to

$$\hat{\mathcal{I}}^{0l} = \frac{\sqrt{-\bar{g}}}{2\kappa a^2} \left[(2\dot{a}\tilde{h}_0^l - \nabla^k q_k^l)\xi^0 + q_k^l \nabla^k \xi^0 + Q_k^l \xi^k + \tilde{h}_{0k} \nabla^{[k} \xi^{l]} \right]. \tag{4.18b}$$

The linearized expression for $\hat{\mathcal{I}}^0$ is much simpler than it appears in Eq. (3.33) because $\tau^{\mu\nu}=0$. In terms of the energy-momentum perturbations $\delta T^{\mu}_{\nu}=T^{\mu}_{\nu}-\bar{T}^{\mu}_{\nu}$ rather than perturbations of densities we find that

$$\hat{\mathcal{I}}^{0} = \frac{\sqrt{-\bar{g}}}{\kappa a^{2}} \left[\kappa a^{2} \delta T_{0}^{0} \xi^{0} + \kappa a^{2} \delta T_{k}^{0} \xi^{k} - \bar{y} \tilde{h}_{n}^{n} + \frac{1}{2} \bar{z} \mathcal{Q} + \nabla_{n} (\frac{1}{4} \bar{z} \tilde{h}_{0}^{n}) \right]. \tag{4.19}$$

Equation (4.19) suggests that it makes sense to transfer $\nabla_n(\frac{1}{4}\bar{z}\tilde{h}_0^n)$ from the left to the right hand side in $\hat{\mathcal{I}}^0 = \partial_l \hat{\mathcal{I}}^{0l}$ and to rewrite it in the following "renormalized" form appropriate to a 3-dimensional formalism:

$$*\mathcal{I} = \nabla_l \left(*\mathcal{I}^l \right), \qquad *\mathcal{I} \equiv \frac{\kappa a^2}{\sqrt{-\bar{g}}} \hat{\mathcal{I}}^0 - \nabla_l \left(\frac{1}{4} \bar{z} \tilde{h}_0^l \right), \quad *\mathcal{I}^l \equiv \frac{\kappa a^2}{\sqrt{-\bar{g}}} \hat{\mathcal{I}}^{0l} - \frac{1}{4} \bar{z} \tilde{h}_0^l. \tag{4.20}$$

The zero indices are no more appropriate because the stared quantities are not the components of conserved vectors or superpotentials anymore. With the definitions in Eq. (4.20) we can write instead of Eq. (4.19)

$$*\mathcal{I} = \Pi_0 \xi^0 + \Pi_k \xi^k - \bar{y}\tilde{h}_n^n + \frac{1}{2}\bar{z}\mathcal{Q}, \qquad \Pi_\mu \equiv \kappa a^2 \delta T_\mu^0, \tag{4.21}$$

and instead of (4.18b)

$$*\mathcal{I}^{l} = \frac{1}{2} \left(-\nabla^{k} q_{k}^{l} \xi^{0} + q_{k}^{l} \nabla^{k} \xi^{0} + Q_{k}^{l} \xi^{k} - \frac{2}{3} \tilde{h}_{0}^{l} \nabla_{k} \xi^{k} + \tilde{h}_{0k} \nabla^{[k} \xi^{l]} \right). \tag{4.22}$$

Integrating $*\mathcal{I} = \nabla_l \left(*\mathcal{I}^l \right)$ over a sphere (r = const) at constant time $\eta = const$ we obtain

$$\int_{\eta} *\mathcal{I}\sqrt{f}d^3x = r^2\sqrt{f} \oint_{r} \delta_{kl} \left(*\mathcal{I}^k\right) \frac{x^l}{r} sin(\theta) d\theta d\phi. \tag{4.23}$$

We now give the list of the $15 *\mathcal{I}$'s and their associated $*\mathcal{I}^l$'s. Some linear combinations with η dependent factors have greater simplicity. Such combinations break of course the group character of the algebra of globally conserved quantities but here we are interested in integral constraints at a given time η for which the group properties of our currents are not important here. We shall keep trace however of the corresponding η -dependent combinations of conformal Killing vectors and use special symbols for $*\mathcal{I}$ and $*\mathcal{I}^l$ that reminds us of their origin. For instance $*\mathcal{I}(\mathbf{t})$ is denoted by \mathcal{T} , $*\mathcal{I}(s_a)$ by \mathcal{S}_a and so on...

Thus for k=0 and $k=\pm 1$ we have the following $*\mathcal{I}$'s and $*\mathcal{I}^l$'s with $*\mathcal{I}=\nabla_l(*\mathcal{I}^l)$:

$$\mathbf{t} \to \mathcal{T} = \nabla_l \mathcal{T}^l, \qquad \mathcal{T} = \Pi_0 + 2\dot{a}\mathcal{Q} - k\tilde{h}_n^n = \mathcal{T}_0 - k\tilde{h}_n^n, \quad \mathcal{T}^l = -\frac{1}{2}\nabla^k q_k^l,$$
 (4.24 *I*)

$$\mathbf{s}_a \to \mathcal{S}_a = \nabla_l \mathcal{S}_a^l, \quad \mathcal{S}_a = \Pi_a \sqrt{1 - kr^2}, \quad \mathcal{S}_a^l = \frac{1}{2} Q_a^l \sqrt{1 - kr^2} - k \tilde{h}_{0k} x^{[k} s_a^{l]}, \quad (4.24 II)$$

$$\mathbf{r}_a \to \mathcal{R}_a = \nabla_l \mathcal{R}_a^l, \quad \mathcal{R}_a = \Pi_k \epsilon_{kan} x^n, \quad \mathcal{R}_a^l = \frac{1}{2} (Q_k^l \epsilon_{kan} x^n + \tilde{h}_{0k} \epsilon^{akl}) - k \tilde{h}_{0k} x^{[k} r_a^{l]}. \tag{4.24 III}$$

The following 8 quantities in which appears \mathcal{T}_0 defined in Eq. (4.24 I) are for k=0 only:

$$\mathbf{l}_a - \eta \mathbf{s}_a \to \mathcal{L}_a = \nabla_l \mathcal{L}_a^l, \quad \mathcal{L}_a = \mathcal{T}_0 x^a, \quad \mathcal{L}_a^l = -\frac{1}{2} \nabla^k q_k^l x^a + \frac{1}{2} q_a^l,$$
 (4.24 *IV*)

$$\mathbf{d} - \eta \mathbf{t} \to \mathcal{D} = \nabla_l \mathcal{D}^l, \quad \mathcal{D} = \mathcal{S}_a x^a + 2\mathcal{Q}, \quad \mathcal{D}^l = \frac{1}{2} Q_k^l x^k - \tilde{h}^{0l},$$
 (4.24 V)

$$\mathbf{a}_{o} + \eta^{2} \mathbf{t} - 2\eta \mathbf{d} \to \mathcal{A}_{0} = \nabla_{l} \mathcal{A}_{0}, \quad \mathcal{A}_{0} = \mathcal{T}_{0} r^{2} - 2\tilde{h}_{n}^{n}, \quad \mathcal{A}_{0}^{l} = -\frac{1}{2} \nabla_{k} q^{kl} r^{2} + q_{k}^{l} x^{k}, \quad (4.24 \ VI)$$

$$\mathbf{a}_{a} + \eta^{2} \mathbf{s}_{a} - 2\eta \mathbf{l}_{a} \to$$

$$\mathcal{A}_{a} = \nabla_{l} \mathcal{A}_{a}^{l}, \ \mathcal{A}_{a} = 2\mathcal{D}x^{a} - \mathcal{S}_{a}r^{2}, \mathcal{A}_{a}^{l} = Q_{k}^{l}(x^{k}x^{a} - \frac{1}{2}r^{2}\delta_{a}^{k}) + 2\tilde{h}_{0k}\left(\delta_{a}^{[k}x^{l]} - \delta^{kl}x^{a}\right). \tag{4.24 VII}$$

The next 8 linear combinations of conformal Killing vectors are for $k = \pm 1$ only. In those formulas the expressions like $\lambda_a(\beta)$ and $\lambda_a(\partial_0\beta)$ represent the factors of β and $\partial_0\beta$ in the conservation law associated with λ_a . Thus

$$\lambda_a(\partial_0 \beta) \to \tilde{\mathcal{L}}_a = \nabla_l \tilde{\mathcal{L}}_a^l, \quad \tilde{\mathcal{L}}_a = \mathcal{T}_0 x^a, \quad \tilde{\mathcal{L}}_a^l = -\frac{1}{2} \nabla^k q_k^l x^a + \frac{1}{2} q_a^l,$$
 (4.24 \tilde{IV})

$$\delta(\partial_0 \beta) \to \tilde{\mathcal{D}} = \nabla_l \tilde{\mathcal{D}}^l, \ \tilde{\mathcal{D}} = \mathcal{S}_a x^a + 2\mathcal{Q}\sqrt{1 - kr^2}, \ \tilde{\mathcal{D}}^l = (\frac{1}{2}Q_k^l x^k - \tilde{h}_0^l)\sqrt{1 - kr^2}, \ (4.24\ \tilde{V})$$

$$\delta(\beta) \to \tilde{\mathcal{A}}_0 = \nabla_l \tilde{\mathcal{A}}_0^l, \ \tilde{\mathcal{A}}_0 = \mathcal{T}_0 \sqrt{1 - kr^2}, \ \tilde{\mathcal{A}}_0^l = -\frac{1}{2} (\nabla_k q^{kl} + k q_k^l x^k) \sqrt{1 - kr^2}, \ (4.24 \ \tilde{V}I)$$

$$\lambda_a(\beta) \to \tilde{\mathcal{A}}_a = \nabla_l \tilde{\mathcal{A}}_a^l, \quad \tilde{\mathcal{A}}_a = \Pi_k f^{ka} - 2kx^a \mathcal{Q}, \quad \tilde{\mathcal{A}}_a^l = \frac{1}{2} Q_a^l \eta + k \tilde{h}_0^l x^a.$$
 (4.24 VII)

We notice that $\tilde{\mathcal{D}}(k=0) = \mathcal{D}$ and $\tilde{\mathcal{D}}^l(k=0) = \mathcal{D}^l$. Also $\tilde{\mathcal{L}}_a(k=0) = \mathcal{L}_a$ and $\tilde{\mathcal{L}}_a^l(k=0) = \mathcal{L}_a$. However, $\tilde{\mathcal{A}}_0(k=0) \neq \mathcal{A}_0$ and $\tilde{\mathcal{A}}_0^l(k=0) \neq \mathcal{A}_0^l$ as well as $\tilde{\mathcal{A}}_a(k=0) \neq \mathcal{A}_a$. and $\tilde{\mathcal{A}}_a^l(k=0) \neq \mathcal{A}_a^l$.

(ii) Analysis of these results.

 \mathcal{T} , \mathcal{S}_a , \mathcal{R}_a and \mathcal{L}_a , \mathcal{D} , \mathcal{A}_0 , \mathcal{A}_a for k = 0 or $\tilde{\mathcal{L}}_a$, $\tilde{\mathcal{D}}$, $\tilde{\mathcal{A}}_0$, $\tilde{\mathcal{A}}_a$ for $k = \pm 1$ contain three types of terms: linear combinations of $a^2 \kappa \delta T_{\mu}^0 = \Pi_{\mu}$, the perturbation of the "uniform Hubble expansion" \mathcal{Q} and the trace of the perturbation of the spatial components of the metric \tilde{h}_n^n .

 S_a and R_a are homogeneous in Π_{μ} . Besides these 6 quantities there are 4 additional linear combinations with η -dependent coefficients that are homogeneous in Π_{μ} . For k=0

$$\dot{a}^{-1}\mathcal{L}_{a} - \frac{1}{2}\mathcal{A}_{a} = \dot{a}^{-1}\Pi_{0}x^{a} - \Pi_{k}(x^{k}x^{a} - \frac{1}{2}\delta^{ka}r^{2}) \equiv \Pi_{\mu}V_{a}^{\mu},$$

$$\dot{a}^{-1}\mathcal{T} - \mathcal{D} = \dot{a}^{-1}\Pi_{0} - \Pi_{k}x^{k} \equiv \Pi_{\mu}V_{0}^{\mu}.$$

$$(k = 0) \qquad (4.25)$$

For $k = \pm 1$

$$\dot{a}^{-1}\tilde{\mathcal{L}}_a + k\tilde{\mathcal{A}}_a = \dot{a}^{-1}\Pi_0 x^a + k\Pi_k f^{ka} \equiv \Pi_\mu \tilde{V}_a^\mu,
\dot{a}^{-1}\tilde{\mathcal{A}}_0 - \tilde{\mathcal{D}} = (\dot{a}^{-1}\Pi_0 - \Pi_k x^k)\sqrt{1 - kr^2} \equiv \Pi_\mu \tilde{V}_0^\mu.$$
(k = ±1) (4.26)

The V's and \tilde{V} 's are Traschen's [1] "integral constraint vectors". Her vectors are thus linear combinations of conformal Killing vectors with time dependent coefficients. In particular the $\tilde{\mathbf{V}}_a$'s are linear combinations of $\lambda_a(\partial_0\beta)$'s and $\lambda_a(\beta)$'s while $\tilde{\mathbf{V}}_0$ is a combination of $\delta(\partial_0\beta)$'s and $\delta(\beta)$.

It is however clear that if we take the uniform Hubble expansion gauge condition

$$Q = 0, (4.27)$$

then 14 of the 15 "conservation laws" have volume integrands that are linear and homogeneous in Π_{μ} . The exception is for the conformal time translation $(k = \pm 1)$ or acceleration (k = 0):

$$\mathcal{T} = \Pi_0 - k\tilde{h}_n^n, \quad (k = \pm 1); \qquad \mathcal{A}_0 = \Pi_0 r^2 - 2\tilde{h}_n^n, \quad (k = 0).$$
 (4.28)

Thus if Q = 0, the conformal Killing vectors provide 14 linearly independent expressions that are momenta of order 0, 1 and 2 and are given by surface integrals involving boundary values only. Such expressions can in principle be constructed directly from Einstein's constraint equations. The constructs are however far from obvious.

(iii) Example: spacetimes that are asymptotically Schwarzschild-de Sitter (k = 0).

In this example, the background is a de Sitter spacetime with k=0 and perturbations far away from its sources appear to be spherically symmetrical. However, perturbations may be large at and near the sources. The asymptotic metric in our coordinates has been given by Bičák and Podolski [43] [their formula (48)]. Neglecting powers of m/r higher than one, their metric is as follows

$$ds^{2} = d\tau^{2} - (e^{2H\tau} + 2F)d\chi^{2} - (e^{2H\tau} - F)\chi^{2}(d\theta^{2} + \sin^{2}(\theta)d\phi^{2}), \quad F \equiv \frac{2m}{\Lambda}\chi^{-3}e^{-H\tau} \quad (4.29)$$

and $H = (\Lambda/3)^{1/2}$; m and Λ are constants. We set

$$\eta = \varepsilon e^{-H\tau}, \quad r = H\chi, \quad \varepsilon \equiv \pm 1$$
(4.30)

so that Eq. (4.29) takes no this form

$$ds^{2} = (H\eta)^{-2} [d\eta^{2} + (-\delta_{kl} + \tilde{h}_{kl}) dx^{k} dx^{l}].$$
(4.31)

Thus, comparing to Eq. (4.13), we see that

$$a = (H|\eta|)^{-1}, \quad \tilde{h}_{00} = 0, \quad \tilde{h}_{0l} = 0, \quad \tilde{h}_{kl} = -\varepsilon \frac{2}{3} m H \left(\frac{\eta}{r}\right)^3 \left(\delta_{kl} - \frac{3\delta_{km} x^m \delta_{ln} x^n}{r^2}\right).$$
 (4.32)

Notice that Q = 0 in this example and furthermore $\tilde{h}_n^n = 0$. To calculate $*\mathcal{I}^l$ we need the q's and Q's defined in Eq. (4.16b): $q_k^l = \tilde{h}_k^l$ and $Q_k^l = -(3/\eta)\tilde{h}_k^l$. With these elements we find that all 13 integrals that follow are zero:

$$\int_{\infty} \mathcal{T}\sqrt{f}d^3x = \int_{\infty} \mathcal{S}_a\sqrt{f}d^3x = \int_{\infty} \mathcal{R}_a\sqrt{f}d^3x = \int_{\infty} \mathcal{L}_a\sqrt{f}d^3x = \int_{\infty} \mathcal{A}_a\sqrt{f}d^3x = 0.$$
(4.33)

The equalities constitute as many Traschen-like integral constraints. The 2 constants of motion that are not equal to zero are associated with dilatations and with time accelerations in Minkowski spacetime which is conformal to our background

$$-\frac{\varepsilon}{\kappa\eta^2} \int_{\infty} \mathcal{D}\sqrt{f} d^3x = HMc^2, \quad \frac{\varepsilon}{\kappa\eta^3} \int_{\infty} \mathcal{A}_0 \sqrt{f} d^3x = \frac{2}{3}HMc^2. \tag{4.34}$$

In these expressions $M = mc^2/G$. Notice that the 15 integrands of Eqs. (4.33) and (4.34) do not have to be linearized. Near the sources perturbations may be large. If the perturbations are weak in the whole space, we may write instead of Eq. (4.34)

$$-\frac{\varepsilon}{\eta^2} \int_{\infty} \delta T_k^0 x^k \sqrt{f} d^3 x = H M c^2, \quad \frac{\varepsilon}{\eta^3} \int_{\infty} \delta T_0^0 r^2 \sqrt{f} d^3 x = \frac{2}{3} H M c^2. \tag{4.35}$$

These are integral constraints on δT_0^0 and δT_k^0 .

5. Comments on the role of superpotentials in the theory of conservation laws

(i) Motivations.

Here we want to connect our work with past literature, give due credit to yet unmentioned papers and make contact with some well known superpotentials or energymomentum tensors that we have not yet encountered.

(ii) On superpotentials in conservation laws today.

Perhaps the single most important legacy of studies on conservation laws in general relativity is that conserved quantities in finite volumes can always be expressed as surface integrals on the boundary of the volume. Anti-symmetric tensor densities like $\hat{\mathcal{I}}^{\mu\nu}$ dominate the scene to day in the literature, not $\hat{\mathcal{I}}^{\mu}$. One great push in that direction was given by Penrose [14] who introduced the notion of "quasi-local" quantities which, in the weak field limit, reduce to ordinary conserved linear and angular momentum of the gravitational field in finite volumes. Many papers have been published on the subject in particular on quasilocal energy. Unfortunately, selection rules are few and no consensus exists. There is an interesting comparison of formulas in a paper by Berqvist [44] on the energy enclosed by the outer horizon of a Kerr black hole in which it is shown how six different formulas give five different results. There exists however a common point to those various definitions of quasi-local energy: they are not related by differential conservation laws to Einstein's equations [45]. In this instance, the present work points in a very different direction. We have not tried to make the connection with our own superpotential. The role and importance of superpotentials in field theory has been emphasized by Julia and Silva [28] who gave them an elegant and general mathematical basis.

(iii) Connection with other superpotentials on a flat background.

Rosen [19] was the first to drive attention to the fact that the quadratic $\Gamma\Gamma$ -Lagrangian, Eq. (2.6) used by Einstein to derive a conserved pseudo-tensor could be written in covariant form by introducing a second metric. This amount in practice to describe curved spacetimes with respect to a flat background. The mathematical basis of Rosen's approach is given in Lichnerowiscz [46]. It is thus no surprise that our formulation of conservation laws for perturbations of curved backgrounds connects nicely with well known conservation laws in classical general relativity. This is what we want to show here. Let us go back for a moment to the "divergence dependent" conserved vector \hat{I}^{μ} with Eq. (2.17), not $\hat{\mathcal{I}}^{\mu}$. The Rosenfeld identities have been worked out in KBL97****. On a flat background, $\overline{R}^{\lambda}_{\nu\rho\sigma} = 0$ and in arbitrary coordinates, the formulas of KBL97 are similar to Eqs. (3.27) - (3.30). In particular,

$$\overline{D}_{\mu}\hat{\theta}^{\mu}_{\nu} = 0, \quad \hat{\theta}^{\mu\nu} = -\overline{D}_{\lambda}\hat{\sigma}^{\lambda\mu\nu}, \quad \hat{\theta}^{\mu}_{\nu} = \hat{T}^{\mu}_{\nu} + \hat{t}^{\mu}_{\nu}, \tag{5.1}$$

 \hat{t}^{μ}_{ν} is Einstein's energy-momentum tensor density as given by Rosen in arbitrary coordinates. On a flat background $\hat{\theta}^{\mu}_{\nu}$ is thus the divergence of a tensor, not an anti-symmetric one and not a two index tensor but still one acting like a "superpotential" for volume integrals in Minkowski coordinates. $\hat{\sigma}^{\lambda\mu\nu}$ is Tolman's [47] superpotential, apparantly the first of its kind in the literature. This superpotential is related to another famous one, Freud's [33] superpotential $\hat{F}^{\lambda\mu\nu}$ is defined in arbitrary coordinates by Eq. (2.22):

$$\hat{\sigma}^{\lambda\mu\nu} = \hat{F}^{\lambda\mu\nu} + \frac{1}{2\kappa} \overline{D}_{\rho} \left(\bar{g}^{\nu[\rho} \hat{l}^{\lambda]\mu} \right). \tag{5.2}$$

Since covariant derivatives on a flat spacetime are commutative, by taking the divergence of $\hat{\sigma}^{\lambda\mu\nu}$ and using its relation with $\hat{\theta}^{\mu\nu}$ - see Eq. (5.1) - we obtain a similar relation

$$\hat{\theta}^{\mu\nu} = -\overline{D}_{\lambda}\hat{F}^{\lambda\mu\nu}.\tag{5.3}$$

The great simplicity of Freud's superpotential made it a successful quantity to calculate globally conserved quantities like the total energy at spatial infinity as well as at null infinity [48]. Our own energy tensor on a flat background satisfies similar relations.

$$\overline{D}_{\mu}\hat{T}^{\mu}_{\nu} = 0, \quad \hat{T}^{\mu\nu} = -\overline{D}_{\lambda}(*\hat{S}^{\lambda\mu\nu}) = -\overline{D}_{\lambda}\hat{\mathcal{P}}^{\lambda\mu\nu}. \tag{5.4}$$

The difference between $\hat{T}^{\mu\nu}$ and $\hat{\theta}^{\mu\nu}$ following from Eq. (3.10) is exactly $\overline{D}_{\rho}\hat{S}^{\rho\mu\nu}$. It is interesting to notice the relation between the divergences of the *S and \mathcal{P} tensors when backgrounds are not flat:

$$\overline{D}_{\lambda}\hat{\mathcal{P}}^{\lambda\nu\mu} = \overline{D}_{\lambda}(*\hat{S}^{\lambda\mu\nu}) - \frac{1}{2\kappa}\hat{l}^{\rho\sigma}\overline{R}^{\mu}_{\rho\sigma}{}^{\nu} - \frac{1}{2\kappa}\hat{l}^{\lambda\nu}\overline{R}^{\mu}_{\lambda}. \tag{5.5}$$

^{****} In KBL97, the formulas are (2.50) to (2.52) but a $\hat{}$ is missing on every symbol!

A most famous superpotential is that of Komar, $(1/\kappa)D^{[\mu}\hat{\xi}^{\nu]}$. Its greatest quality is to be background independent. It is also useful in asymptotically flat spacetimes at spatial infinity. But it has some shortcomings which we already mentioned. There have been various corrections of that attractive covariant expression [49] which did not get rid of the anomalous factor 2 and had also other "defects" [45].

One intriguing superpotential is that of Arnowitt, Deser and Misner [50], especially for energy. It was original defined in a synchronous gauge, at least asymptotically, $g_{00} = 1$, $g_{0k} = 0$. In that gauge, the surface integral at spatial infinity of the Komar tensor is zero. What remains then of the superpotential in Eq. (2.21) is the (0k)-component of $(1/\kappa)\xi^{[\mu}\hat{k}^{\nu]}$ which reduce indeed to the ADM integrand at infinity as can easily be verified.

The Landau and Lifshitz superpotential is as satisfactory as our own superpotential for calculating the total 4-linear momentum but it has the wrong weight and it is difficult to see how to connect it with the group of diffeomorphisms via Noether's method on a curved background. The L-L complex has however been obtained recently from a variational principle by Babak and Grishchuk [51] on a flat background in arbitrary coordinates. Incidentally L-L's pseudo-tensor has one more drawback, not shared by Einstein's pseudo-tensor which was pointed out by Chandrasekhar and Ferrari [52]. Consider the weak field approximation of the total energy in a stationary spacetime. A variational principle applied to the total "Einstein Energy" leads to Einstein's linearized field equations. The "L-L Energy" provides incorrect equations.

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Appendix on Globally Conserved Quantities

(i) Object of this appendix.

There exists quite a number of different conservation laws but few criteria to select among them. The main discriminating conditions are global conservation laws and the weak field limit which are to some extend related. The only non-ambiguously defined global quantities are the 4 components of the linear momentum P_{α} at spatial and at null infinity for spacetimes with definite fall off conditions to asymptotic flatness. It is therefore important to show that our superpotential gives at least the correct total 4-momentum in those cases. In this task we shall be greatly helped by Eq. (3.11) which for Killing vector is

$$\hat{\mathcal{J}}^{\mu\nu} = \hat{J}^{\mu\nu} + \hat{S}^{\mu\nu\rho}\bar{\xi}_{\rho}. \tag{A.1}$$

It has been shown that the superpotential $\hat{J}^{\mu\nu}$ provides by itself the total 4-momentum P_{α} both at spatial infinity [16] and at null infinity [53]. We must therefore show that $\hat{S}^{\mu\nu\rho}\bar{\xi}_{\rho}$ does not contribute to P_{α} in both asymptotic directions. This is what we briefly indicate in this appendix. We shall show that in both asymptotic directions the "discrepancy"

$$\Delta P_{\alpha} = \oint_{S} \hat{S}^{\mu\nu\rho} \bar{\xi}_{(\alpha)\rho} dS_{\mu\nu} = 0. \tag{A.2}$$

S is the sphere at infinity and $\bar{\xi}_{(\alpha)\rho}$, ($\alpha = 0, 1, 2, 3$) are the four Killing vectors of translations in the asymptotically flat background.

(ii) Stationary spacetimes and spatial infinity.

Consider stationary solutions that fall off as follows in asymptotic Minkowski coordinates at spatial infinity $x^{\mu} = (x^0 = t, x^k)$:

$$g_{\mu\nu}(x^k) = \eta_{\mu\nu} + \frac{1}{r}u_{\mu\nu} + O_2, \qquad \partial_l g_{\mu\nu}(x^k) = \frac{1}{r^2}v_{l\mu\nu} + O_3$$
 (A.3)

where $r = \sqrt{\Sigma(x^k)^2}$ and $u_{\mu\nu}$, $v_{l\mu\nu}$ depend on the directions on the sphere at infinity. The background metric is $\eta_{\mu\nu}$ and the Killing vector of translations $\bar{\xi}_{(\alpha)\nu} = \eta_{\alpha\nu}$. The discrepancy is therefore given by

$$\Delta P_{\alpha} = \oint_{r \to \infty} \hat{S}^{0l\alpha} n_l \ r^2 sin\theta d\theta d\phi, \qquad n_l = \frac{X^l}{r}. \tag{A.4}$$

It takes some patience to make this calculation, using the asymptotic form of the metric (A.3) in Eq. (3.8) with Eq. (3.21), but the calculation does not need clever tricks and we find indeed that $\Delta P_{\alpha} = 0$ for $r \to \infty$. The metric does not have to fall off as fast as in Eq. (A.3). $\Delta P_{\alpha} = 0$ under weaker fall off conditions has been studied in [54].

(iii) Radiation superpotentials at null infinity.

For the Bondi-Sachs metric we use the Newman and Unti [55] representation with coordinates $x^{\lambda} = (x^0 = u, x^1 = r, x^2, x^3)$ in which the metric has the following form

$$ds^{2} = g_{00}du^{2} + 2dudr + 2g_{0L}dudx^{L} + g_{KL}dx^{K}dx^{L}$$
(A.5)

and the flat background has a

$$d\bar{s}^2 = du^2 + 2dudr - \frac{r^2}{2P^2}[(dx^2)^2 + (dx^3)^2], \qquad P = \frac{1}{2} + \frac{1}{4}[(x^2)^2 + (x^3)^2]. \tag{A.6}$$

The asymptotic form of the metric depends on 5 independent real functions of (u, x^2, x^3) , namely ψ'_1 , ψ''_1 , ψ''_2 , σ' and σ'' . These notations are similar to those of Newman and Unti-without a zero index - who use complex functions; here the prime indicates the real part, two primes indicate the complex part of their complex functions. The asymptotic form of the metric in these notation is

$$g_{00} = 1 + \frac{2\psi_2'}{r} + O_2, \tag{A.7}$$

$$g_{02} = -P^2 \left[\partial_2 \left(\frac{\sigma'}{P^2}\right) + \partial_3 \left(\frac{\sigma''}{P^2}\right)\right] + \frac{2}{3} \frac{\psi'_1}{Pr} + O_2, \quad g_{03} = P^2 \left[\partial_3 \left(\frac{\sigma''}{P^2}\right) - \partial_2 \left(\frac{\sigma'}{P^2}\right)\right] + \frac{2}{3} \frac{\psi''_1}{Pr} + O_2, \quad (A.8)$$

$$g_{23} = -r\frac{\sigma''}{P^2} + O_1, \quad g_{22} = -\frac{r^2}{2P^2} - \frac{r\sigma'}{P^2} + O_0, \quad g_{33} = -\frac{r^2}{2P^2} + \frac{r\sigma'}{P^2} + O_0.$$
 (A.9)

with $|\sigma|^2 = \sigma'^2 + \sigma''^2$.

The Killing vector of translations in the flat background spacetime with the metric (A.6) have the following components

$$\bar{\xi}_{(0)\mu} = (1, 1, 0, 0), \quad \bar{\xi}_{(m)\mu} = (0, -n_m, -r\partial_2 n_m, -r\partial_3 n_m), \quad n_m = \frac{X^m}{r}.$$
 (A.10)

With Eq. (A.5) to Eq. (A.10), we have the necessary elements to calculate Eq. (A.2) at null infinity using Eq. (3.8) with Eq. (3.21). The calculation is even more tedious than before but still it is not difficult to show that indeed $\Delta P_{\alpha} = 0$ for $r \to \infty$.

The reader may remember that the loss of energy E per unit u-time is given by

$$\frac{dE}{du} = -(8\pi)^{-1} \oint [(\partial_u \sigma')^2 + (\partial_u \sigma'')^2] P^{-2} dx^2 dx^3 < 0. \tag{A.11}$$

This formula is one of the outstanding results of Bondi.

The loss of energy obtained from the Abbot and Deser [37] superpotential is different and is not negative definite. Following the prescription indicated in section 3 this superpotential satisfy the following expression

$$\kappa J_{AD}^{\mu\nu} = -H^{\rho[\mu} \overline{D}_{\rho} \bar{\xi}^{\nu]} + \bar{\xi}^{\rho} \overline{D}^{[\mu} H_{\rho}^{\nu]} - \bar{\xi}^{[\mu} \overline{D}_{\rho} H^{\nu]\rho} \tag{A.12}$$

where

$$H_{\mu\nu} = *h_{\mu\nu} - \frac{1}{2}\overline{g}_{\mu\nu}(*h), \quad *h_{\mu\nu} = g_{\mu\nu} - \overline{g}_{\mu\nu}, \quad *h = \overline{g}^{\rho\sigma}(*h_{\rho\sigma}).$$
 (A.13)

If we use this superpotential to calculate the energy E_{AD} and the corresponding energy loss, we find that

$$\frac{dE_{AD}}{du} = \frac{dE}{du} + (8\pi)^{-1} \frac{d^2}{du^2} \oint \frac{|\sigma|^2}{P^2} dx^2 dx^3.$$
 (A.14)

This is not negative definite. The Abbott and Deser superpotential should be rejected on this basis.

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